

TOPOLOGICAL T-DUALITY, AUTOMORPHISMS AND CLASSIFYING SPACES

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ABSTRACT. We extend the formalism of Topological T-duality to spaces which are the total space of a principal S^1 -bundle $p : E \rightarrow W$ with an H -flux in $H^3(E, \mathbb{Z})$ together with an automorphism of the continuous-trace algebra on E determined by H . The automorphism is a ‘topological approximation’ to a gerby gauge transformation of spacetime. We motivate this physically from Buscher’s Rules for T-duality. Using the Equivariant Brauer Group, we connect this problem to the C^* -algebraic formalism of Topological T-duality of Mathai and Rosenberg [1].

We show that the study of this problem leads to the study of a purely topological problem, namely, Topological T-duality of triples (p, b, H) consisting of isomorphism classes of a principal circle bundle $p : X \rightarrow B$ and classes $b \in H^2(X, \mathbb{Z})$ and $H \in H^3(X, \mathbb{Z})$. We construct a classifying space $R_{3,2}$ for triples in a manner similar to the work of Bunke and Schick [2]. We characterize $R_{3,2}$ up to homotopy and study some of its properties. We show that it possesses a natural self-map which induces T-duality for triples. We study some properties of this map.

1. INTRODUCTION

Topological T-duality is an attempt to study the T-duality symmetry of Type II String theory [3] using methods from Noncommutative and Algebraic Topology [1, 2, 4, 5]. In the simplest case, T-duality states that Type II A string theory on a certain background is equivalent to Type II B string theory on another background, that is, acting on the original string theory by a canonical transformation (termed T-duality) transforms that theory into the dual one. For this to be possible, the background spacetime must carry a torus action (which need not be free). In this paper we only consider backgrounds which are principal circle bundles.

Both Type IIA and Type IIB String Theory backgrounds possess three massless bosonic fields: A graviton (associated to the metric), an H -flux (associated to the Kalb-Ramond field B , $H = dB$) and a

dilaton. As is well known [6], the H -flux is a closed integral three-form which is the gerbe curvature of a gerbe with connection form B on that background (The word ‘gerbe’ is used in the sense of the paper Ref. ([6]).) It is a remarkable fact that the Topology and H -flux of the T-dual spacetime depend only on the Topology and H -flux of the original spacetime¹. This phenomenon is called Topological T-duality.

Let X be the manifold which is the model for spacetime. Let $\{U_\alpha\}$ be an open cover of X . We assume that X is a principal circle bundle over a base W . We perform a T-duality along the circle orbits in X . It is clear that the T-dual background will contain the Topological T-dual $X^\#$ (in the sense of Refs. [1, 2, 4, 5]) as a factor.

Consider a gerbe with connection on X which is equivariant under the S^1 -action on X and let $B_\alpha = B|_{U_\alpha}$ be the gerbe connection restricted to U_α and $H = dB_\alpha$ the gerbe curvature form. In this paper we study large gauge transformations of the gerbe which are equivariant under the circle action on X .

A large gauge transformation may be obtained by tensoring the gerbe on X with a line bundle $L \rightarrow X$ with connection (See Ref. [6] after Eq. (2.10), for example.). Further, the large gauge transformations generate the gauge group. Locally such transformations are of the form $B_\alpha \rightarrow B_\alpha + F|_{U_\alpha}$ where F is the curvature of a line bundle with connection on X . We require that L be equivariant under the circle action on X . Such a gauge transformation has as a characteristic class the integral cohomology class of F , namely $[F] \in H^2(X, \mathbb{Z})$. Also, due to equivariance, (see Ref. [6] Appendix C for a treatment of equivariant line bundles on a principal bundle in another context) the Lie derivative of F with respect to the circle action vanishes.

We write the metric and B -field on X in Kaluza-Klein form. We choose coordinates (θ, x_α) locally on the cover $\{U_\alpha\}$. We write the metric and B -field on X in Kaluza-Klein form as

$$\begin{aligned} ds^2 &= G_{00}(d\theta + A_\alpha dx^\alpha)^2 + g_{\alpha\beta} dx^\alpha dx^\beta \\ B &= (d\theta + A_\alpha) \wedge B_\beta dx^\beta + B_{\alpha\beta} dx^\alpha \wedge dx^\beta. \end{aligned}$$

¹See Refs. [1, 2, 4, 5] for a variety of approaches.

If we denote T-dual quantities by tilde superscripts, Buscher's rules take the form

$$\begin{aligned}\tilde{G}_{00} &= 1/G_{00}, \\ \tilde{A}_\alpha &= B_\alpha, \quad \tilde{B}_\alpha = A_\alpha, \\ \tilde{g}_{\alpha\beta} &= g_{\alpha\beta}, \quad \tilde{B}_{\alpha\beta} = B_{\alpha\beta}, \\ \tilde{\Phi} &= \Phi - \frac{1}{2} \ln(G_{00})\end{aligned}$$

(See, for example, Ref. [7], Ex. (6.12)).

Let Θ denote the connection form on the bundle X and $\tilde{\Theta}$ the connection form on the T-dual bundle. From the above

$$\begin{aligned}\Theta &= d\theta + A_\alpha dx^\alpha \\ \tilde{\Theta} &= d\tilde{\theta} + B_\alpha dx^\alpha\end{aligned}$$

where $\Theta = d\theta + A_\alpha dx^\alpha$ and $\tilde{\Theta} = d\tilde{\theta} + B_\alpha dx^\alpha$ and we have used Buscher's Rules above. Now, suppose $B \rightarrow B + F$ on X where $dF = 0$ and F is integral. We may write $F = \Theta \wedge F_1 + F_2$ where F_i are horizontal forms on X . The T-dual B -field and metric will now be

$$\begin{aligned}ds^2 &= G_{00}(d\tilde{\theta} + B_\alpha dx^\alpha + F_{1\alpha} dx^\alpha)^2 + g_{\alpha\beta} dx^\alpha dx^\beta \\ \tilde{B} &= (d\tilde{\theta} + B_\alpha dx^\alpha + F_{1\alpha} dx^\alpha) \wedge A_\beta dx^\beta + \\ &\quad B_{\alpha\beta} dx^\alpha \wedge dx^\beta + F_{2\alpha\beta} dx^\alpha \wedge dx^\beta.\end{aligned}$$

It is easy to see that since $dF = 0$, $dF_1 = 0$ and, on the T-dual the form dF_1 is zero as well. However F_2 is not a closed form in general.

It can be seen directly from the definition that $F_1 = p_!(F)$, and so, since F is integral, F_1 is integral as well.

We may now make a coordinate transformation of $X^\#$ which will get rid of F_1 . Since the gerbe connection on X and on the T-dual are both equivariant under the circle action, making such a coordinate transformation will cause the T-dual B -field to undergo a gauge transformation. This can be seen directly as follows: It is clear that this coordinate transformation will be a bundle automorphism (denoted ϕ) of $X^\#$ and so $q \circ \phi = q$. Recall that $B_\alpha - B_\beta = dA_{\alpha\beta}$ by definition. We have that

$$\phi^*(B_\alpha) - \phi^*(B_\beta) = \phi^*(dA_{\alpha\beta}) = d\phi^*A_{\alpha\beta}.$$

However, by equivariance²

$$A_{\alpha\beta} = q^*a_{\alpha\beta} + h_{\alpha\beta}\Theta^\#$$

²See for example Ref. [6], Eq. (2.14).

where $a_{\alpha\beta}, h_{\alpha\beta}$ are horizontal one and zero forms on $X^\#$ (See For eg. Ref. [6] Eq. 2.14). Therefore,

$$\phi^* A_{\alpha\beta} = \phi^* a_{\alpha\beta} + \phi^*(h_{\alpha\beta})\Theta^\#$$

(Here $\phi^*\Theta^\# = \Theta^\#$ since ϕ is a bundle automorphism.) Now, ϕ^* will in general, act nontrivially on $h_{\alpha\beta}$ if the bundle transformation ϕ was topologically nontrivial. This implies that, compared to the case when $F = 0$, the T-dual B -field has undergone a large gauge transformation.

As we had said earlier, a large gauge transformation of a gerbe is associated to a class in $H^2(X, \mathbb{Z})$. We may pick a large gauge transformation, and ask for the characteristic class of the gerby gauge transformation on the T-dual.

It was postulated in Ref. [1] that an exterior equivalence class of C^* -dynamical systems of the form $(\mathcal{A}, \mathbb{R}, \alpha)$ should be naturally associated to a spacetime with H -flux which also has a free S^1 -action. The continuous-trace algebra together with its associated \mathbb{R} -action should be viewed as a ‘topological approximation’ to the smooth gerbe on spacetime together with the action of S^1 on it. In Ref. [8] such a C^* -dynamical system was constructed naturally from the data of a smooth gerbe with connection on X in a large variety of examples.

Further, Ref. [1] demonstrated that the effect of the T-duality transformation on spacetime topology was given by the crossed product[9] construction in C^* -algebra theory in all the examples examined: The topological space underlying the T-dual spacetime was always the spectrum of the crossed-product algebra $\mathcal{A} \rtimes_{\alpha} \mathbb{R}$. Based on this, Ref. [1] argued that it was natural to associate to the T-dual the C^* -dynamical system $(\mathcal{A} \rtimes_{\alpha} \mathbb{R}, \hat{\mathbb{R}}, \alpha^\#)$.

We may use the definition of Ref. [1] and take as a model for a spacetime X with H -flux, a continuous-trace algebra \mathcal{A} with spectrum X together with a lift α of the S^1 -action on X to a \mathbb{R} -action on \mathcal{A} [10]. Let \mathcal{K} denote the compact operators on a fixed separable infinite dimensional Hilbert space. Let $p : X \rightarrow W$ be a principal circle bundle and $\mathcal{A} = CT(X, \delta)$, $\delta \in H^3(X, \mathbb{Z})$. Let α be a lift of the S^1 -action on X to an \mathbb{R} -action on \mathcal{A} . Let $\phi : \mathcal{A} \rightarrow \mathcal{A}$ be a locally unitary automorphism commuting with the \mathbb{R} -action. It is well known that such automorphisms are determined up to unitary equivalence by their Phillips-Raeburn obstruction [11] in $H^2(X, \mathbb{Z})$.

The following lemma is based on Thm (3.1) of Ref. [12]:

Lemma 1.1. *For X a principal bundle $p : X \rightarrow W$, Thm. (3.1) of Ref. [12] applies even when $\mathcal{A} = CT(X, \delta)$ for any $\delta \in H^3(X, \mathbb{Z})$.*

Proof. The proof of the Theorem only uses the fact that \mathcal{A} is a continuous-trace algebra (See Ref.[12]) and references therein). The fact that \mathcal{A} is $C_0(X, \mathcal{K})$ is not used anywhere in that theorem. Thus it applies even when the Dixmier-Douady invariant is nonzero. \square

As a model for a gauge transformation of the gerbe associated to \mathcal{A} , we take a spectrum-fixing C^* -algebra automorphism ϕ on \mathcal{A} which commutes with α . We consider the C^* -dynamical system. $(\mathcal{A}, \alpha \times \phi, \mathbb{R} \times \mathbb{Z})$. The C^* -algebra automorphism ϕ then defines an automorphism ψ of the crossed-product algebra (See Ref. [12] and Lemma (1.1) above). What is the Phillips-Raebrun obstruction of this automorphism? We partially answer this question in general in this paper and answer it completely for a large class of examples.

Thus, given any class $[\lambda] \in H^2(X; \mathbb{Z})$, there exists a (not necessarily unique) action α of \mathbb{R} on \mathcal{A} inducing the given action of $S^1 = \mathbb{R}/\mathbb{Z}$ on W and a commuting action λ of \mathbb{Z} on \mathcal{A} with Phillips-Raebrun obstruction $[\lambda]$. Also λ passes to a locally unitary action on $\mathcal{E} \simeq \mathcal{A} \rtimes_{\alpha} \mathbb{R}$.

The actions α and λ are (individually) unique up to exterior equivalence, but unfortunately the pair (α, λ) , as an action of $\mathbb{R} \times \mathbb{Z}$, is not necessarily unique as a C^* -dynamical system, so this construction is not entirely canonical. However, this nonuniqueness does **not** change the Phillips-Raebrun invariant of the T-dual automorphism. Thus the question asked above is well-defined.

In Section (2), we introduce a point of view based on the equivariant Brauer group, which measures precisely this lack of canonicity described above. We show that there is a natural³ map, T_R , which sends a C^* -dynamical system $(\mathcal{A}, \mathbb{R} \times \mathbb{Z}, \alpha \times \phi)$ to a dual⁴ dynamical system $(\mathcal{A} \rtimes_{\alpha} \mathbb{R}, \mathbb{R} \times \mathbb{Z}, \alpha^{\#} \times \phi^{\#})$. To a C^* -dynamical system $(\mathcal{A}, \mathbb{R} \times \mathbb{Z}, \alpha \times \phi)$ we associate a triple

($\hat{\mathcal{A}}$, Phillips-Raebrun invariant of ϕ , Dixmier-Douady invariant of \mathcal{A}).

We show that there is a well-defined map of triples, $T_{3,2}$, which sends a triple

(principal bundle $p : X \rightarrow B$, $b \in H^2(X, \mathbb{Z})$, $\delta \in H^3(X, \mathbb{Z})$)

to another such triple which commutes with the map T_R described above.

³This does not contradict the statement in the previous paragraph, as there we were trying to lift a class in $H^2(X, \mathbb{Z})$ to such a dynamical system.

⁴We are using a different definition of dual dynamical system from Schneider's work, [13], see Sec. (2).

We then attempt to give an answer to the question above. In Sections (3-6), we study some natural properties of the map $T_{3,2}$ and show that owing to the topological properties of this map, when the Dixmier-Douady invariant of \mathcal{A} is fixed, there is a partition of $H^2(X, \mathbb{Z})$ and $H^2(X^\#, \mathbb{Z})$ into cosets such that the T-dual of an automorphism with Phillips-Raeburn invariant in a given coset is an automorphism with Phillips-Raeburn invariant in another coset. This is enough to answer the question posed above for several spaces.

2. A MODEL FOR THE H -FLUX

Let X be a model for spacetime with H -flux H as in Sec. (1). Let U_α be an open cover of X . Then, the H -flux is the curvature of a smooth S^1 -equivariant gerbe on X and $H = dB_\alpha$ locally where B_α is the gerbe connection form in U_α . Changes in the B -field keeping H fixed correspond to acting on the gerbe on spacetime with a gerby gauge transformation. The gauge group of the gerbe is generated by the group of line bundles on X with connection (See Ref. [6], Sec. 2, for example.). That is, if $p : L \rightarrow X$ is a line bundle with curvature two-form F , then, under such a gauge transformation on each patch U_α , $B'_\alpha = B_\alpha + F_\alpha$. That is, $H = dB$ locally, and after such a gauge transformation $H = dB'$ with $d(B - B') = 0$ that is, $(B - B')$ is closed and integral. Since X possesses a S^1 -action (it is the total space of a principal circle bundle with $W = X/S^1$), we require that L is S^1 -equivariant.

As explained in the previous section, in the C^* -algebraic theory of Topological T-duality, the gerbe on spacetime is replaced by a ‘topological approximation’, a continuous-trace algebra \mathcal{A} with spectrum X and Dixmier-Douady invariant equal to H . In this theory, automorphisms of \mathcal{A} define a cohomology class in $H^2(X, \mathbb{Z})$.

A gerby gauge transformation of the type described above naturally defines an automorphism ϕ of the associated continuous-trace algebra with the Phillips-Raeburn class of the automorphism equal to the characteristic class of the line bundle L above. To study arbitrary changes in the B -field, that is to study a general gerby gauge transformation would probably require the introduction of a smooth structure and would be difficult to do in the C^* -algebraic picture of Topological T-duality. However, the theory of integral changes of the B -field is still quite interesting mathematically, as the following sections show.

The above automorphism ϕ of \mathcal{A} gives a \mathbb{Z} -action on \mathcal{A} . However, \mathcal{A} possesses a \mathbb{R} -action already which is a lift of the S^1 -action on X to

\mathcal{A} . Since $L \rightarrow X$ is an equivariant line bundle, ϕ commutes with the \mathbb{R} -action on \mathcal{A} and hence we obtain a $\mathbb{Z} \times \mathbb{R}$ -action on \mathcal{A} .

It will be useful to recall the notion of the Equivariant Brauer Group⁵: Let X be a second countable, locally compact, Hausdorff topological space and let G be a second countable, locally compact, Hausdorff topological group acting on X . Let $\mathfrak{Br}_G(X)$ denote the set of equivalence classes of C^* -dynamical systems consisting of continuous-trace algebras \mathcal{A} on W together with a lift α of the G -action on X to \mathcal{A} . We say that the dynamical system (\mathcal{A}, α) is equivalent to (\mathcal{B}, β) if there is a Morita equivalence bimodule ${}_A M_B$ together with a strongly continuous G -action by linear transformations ϕ_s on M such that for every $s \in G$, $\alpha_s(\langle x, y \rangle_A) = \langle \phi_s(x), \phi_s(y) \rangle_A$, and $\beta_s(\langle x, y \rangle_B) = \langle \phi_s(x), \phi_s(y) \rangle_B$. (Recall, the image of the inner product $\langle, \rangle_A : M \times M \rightarrow \mathcal{A}$ is dense in \mathcal{A} and similarly for \langle, \rangle_B .) It is shown in Ref. [14] that this is an equivalence relation and that the quotient is a *group*. This group is termed the Equivariant Brauer Group $\text{Br}_G(X)$ of X . The group operation is the $C_0(X)$ -tensor product of continuous-trace algebras and group actions.

Recall that an automorphism of a continuous-trace algebra \mathcal{A} can be lifted to a $\mathbb{Z} \times \mathbb{R}$ -action on \mathcal{A} . Now, a $\mathbb{R} \times \mathbb{Z}$ -action on \mathcal{A} gives rise to an element of $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$. The \mathbb{R} -action on \mathcal{A} factors through the S^1 -action on X , while the \mathbb{Z} -action on X is trivial. Also, the lift to the $\mathbb{R} \times \mathbb{Z}$ -action is not unique, as we really only care about the restriction of the action to the \mathbb{R} factor and to the \mathbb{Z} factor up to exterior equivalence. Hence it is really only the class of the action in $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)/H_M^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$ that we need to answer the question in Sec. (1).

Note that elements of $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ would consist of a Morita equivalence bimodule between $\mathbb{R} \times \mathbb{Z}$ -dynamical systems and an \mathbb{R} -equivariant automorphism of the module compatible with the $\mathbb{Z} \times \mathbb{R}$ -action on the dynamical systems. Thus by studying $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ we are simply adding more structure to $\text{Br}_{\mathbb{R}}(X)$.

We conjecture that elements of this group are a good model for spacetimes with a possibly nonzero B -field. Let $F_1 : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \rightarrow \text{Br}_{\mathbb{Z}}(X)$ and $F : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \rightarrow \text{Br}_{\mathbb{R}}(X)$ be the forgetful maps. We formalize the above discussion in the following

Def 2.1. *Let X be a locally compact, finite dimensional CW-complex homotopy equivalent to a finite CW-complex. Let X also be a free S^1 -space with $W = X/S^1$, so that we have a principal S^1 -bundle $p : X \rightarrow$*

⁵We use Ref. [14] here.

W. An element⁶ $y = [\mathcal{A}, \alpha \times \phi]$ of $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ is now defined to be a model for a space with nonzero H -flux or zero H -flux and a nonzero integral B -field. The H -flux H is⁷ $H = F(y) = [\mathcal{A}]$. If $H = 0$, the B -field is the unique class⁸ in $H^2(X, \mathbb{Z})$ which determines $F_1(y)$. This class is equal to the Phillips-Raeburn class of ϕ .

By Ref. [14], there is a natural filtration of $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ given by $0 < B_1 < \ker(F) < \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$, where B_1 is a quotient of $H_M^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$. We argue below that each step in this filtration corresponds to one of the gauge fields in the problem.

We need the following

Theorem 2.1. *Let $p : X \rightarrow W$ be as above.*

- (1) *We have a split short exact sequence*

$$0 \rightarrow \ker(F) \rightarrow \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \xrightarrow{F} \text{Br}(X) \rightarrow 0$$

where $F : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \rightarrow \text{Br}(X)$ is the forgetful map.

- (2) *We have a surjective map $\eta : \ker(F) \rightarrow H^2(X, \mathbb{Z})$.*
(3) *We have a natural isomorphism $H_M^1(\mathbb{R}, C(X, \mathbb{T})_0) \simeq C(W, \mathbb{R})$.*
(4) *The group $H_M^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$ is connected and there is a natural surjective map $q : H_M^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) \rightarrow C(W, \mathbb{T})_0$.*

Proof. (1) We have a forgetful homomorphism $F_1 : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \rightarrow \text{Br}_{\mathbb{R}}(X)$, where $F_1 : [\mathcal{A}, \alpha \times \phi] \rightarrow [\mathcal{A}, \alpha]$. This map is obviously surjective, since we have a section $s : [\mathcal{A}, \alpha] \rightarrow [\mathcal{A}, \alpha \times \text{id}]$.

Since $\text{Br}_{\mathbb{R}}(X) = H^3(X, \mathbb{Z})$ (by Sec. (6.1) of Ref. [14]), the kernel of F_1 consists of Morita equivalence classes of dynamical systems $[\mathcal{A}, \alpha \times \phi]$ such that $\delta(\mathcal{A}) = 0$. Thus, it actually consists of the group $\ker(F)$, where $F : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \rightarrow \text{Br}(X)$ is the map forgetting the group action.

- (2) By Thm. (5.1) of Ref. [14], we have a homomorphism $\eta : \ker(F) \rightarrow H_M^1(\mathbb{R} \times \mathbb{Z}, H^2(X, \mathbb{Z}))$. Now, by Thm. (4.2) of Ref. [1], we have that $H_M^1(\mathbb{R} \times \mathbb{Z}, M) \simeq H^1(B(\mathbb{R} \times \mathbb{Z}), M)$ for any discrete $\mathbb{R} \times \mathbb{Z}$ module M . Also, $B(\mathbb{R} \times \mathbb{Z}) \simeq S^1$, so $H_M^1(\mathbb{R} \times \mathbb{Z}, H^2(X, \mathbb{Z})) \simeq H^2(X, \mathbb{Z})$.

By Thm. (5.1) item (2) of [14], the image of η has range which is all of $H^2(X, \mathbb{Z})$ since $H_M^3(\mathbb{R} \times \mathbb{Z}, C(X, S^1)) = 0$ by Thm. (2.2.1). Hence we have a surjective homomorphism $\eta : \ker(F) \rightarrow H^2(X, \mathbb{Z})$.

⁶Here α is a lift of the S^1 -action on X to a \mathbb{R} -action on \mathcal{A} , while ϕ is a commuting spectrum-fixing \mathbb{Z} -action on \mathcal{A} .

⁷Recall that the forgetful map $\text{Br}_{\mathbb{R}}(X) \rightarrow \text{Br}(X) \simeq H^3(X, \mathbb{Z})$ is an isomorphism [14].

⁸Note that $\text{Br}_{\mathbb{Z}}(X) \simeq H^3(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$.

- (3) We have the following short exact sequence of \mathbb{R} -modules

$$0 \rightarrow H^0(X, \mathbb{Z}) \rightarrow C(X, \mathbb{R}) \rightarrow C(X, \mathbb{T})_0 \rightarrow 0$$

From the associated long exact sequence for H_M^* , we find that $H_M^i(\mathbb{R}, H^0(X, \mathbb{Z})) \simeq 0$, $i = 1, 2$ by Cor. (4.3) of Ref. [1]; hence $H_M^1(\mathbb{R}, C(X, \mathbb{T})_0) \simeq H_M^1(\mathbb{R}, C(X, \mathbb{R}))$. By Thms. 4.5, 4.6, 4.7 of Ref. [1], $H_M^1(\mathbb{R}, C(X, \mathbb{R})) \simeq H_{\text{Lie}}^1(\mathbb{R}, C(X, \mathbb{R})_\infty)$. Here $C(X, \mathbb{R})_\infty$ are the C^∞ -vectors for the \mathbb{R} -action on $C(X, \mathbb{R})$ and so are the functions which are smooth along the S^1 -orbits.

The complex computing the Lie algebra cohomology of \mathbb{R} shows that this group is exactly the functions in $C(X, \mathbb{R})_\infty$ modulo derivatives of functions in $C(X, \mathbb{R})_\infty$ by the generator of the \mathbb{R} -action.

This group is isomorphic to $C(W, \mathbb{R})$ via the ‘averaging’ map $f \rightarrow \int_{S^1} \phi_t \circ f dt$ where $\phi_t \circ f$ is f shifted by the S^1 -action on X .

- (4) We use the spectral sequence calculation of Ref. [12], Thm. (3.1) to note that this group is isomorphic to $H_M^1(\mathbb{R}, H_M^1(\mathbb{Z}, C(X, \mathbb{T})))$. Since \mathbb{Z} is discrete and acts trivially on $C(X, \mathbb{T})$, we have $H_M^1(\mathbb{Z}, C(X, \mathbb{T})) \simeq H^1(\mathbb{Z}, C(X, \mathbb{T})) \simeq C(X, \mathbb{T})$. Hence we need to calculate $H_M^1(\mathbb{R}, C(X, \mathbb{T}))$.

We have the following short exact sequence of \mathbb{R} -modules

$$0 \rightarrow C(X, \mathbb{T})_0 \rightarrow C(X, \mathbb{T}) \rightarrow H^1(X, \mathbb{Z}) \rightarrow 0$$

where $C(X, \mathbb{T})_0$ is the connected component of $C(X, \mathbb{T})$ containing the constant maps.

This gives us a long exact sequence

$$\begin{aligned} H_M^0(\mathbb{R}, C(X, \mathbb{T})) &\rightarrow H_M^0(\mathbb{R}, H^1(X, \mathbb{Z})) \rightarrow H_M^1(\mathbb{R}, C(X, \mathbb{T})_0) \rightarrow \\ &H_M^1(\mathbb{R}, C(X, \mathbb{T})) \rightarrow H_M^1(\mathbb{R}, H^1(X, \mathbb{Z})) \rightarrow \dots \end{aligned}$$

(1)

Also, by Cor. (4.3) of Ref. [1], we have that $H_M^1(\mathbb{R}, H^1(X, \mathbb{Z})) \simeq 0$. Again, by Cor. (4.3) of Ref. [1], we find that $H_M^0(\mathbb{R}, H^1(X, \mathbb{Z})) \simeq H^1(X, \mathbb{Z})$ (since $B\mathbb{R}$ is contractible). Also $H_M^0(\mathbb{R}, C(X, \mathbb{T}))$ consists of the \mathbb{R} -invariant functions in $C(X, \mathbb{T})$ and hence is naturally isomorphic to $C(W, \mathbb{T})$.

Hence we find an exact sequence

(2)

$$C(W, \mathbb{T}) \rightarrow H^1(X, \mathbb{Z}) \rightarrow H_M^1(\mathbb{R}, C(X, \mathbb{T})_0) \rightarrow H_M^1(\mathbb{R}, C(X, \mathbb{T})) \rightarrow 0.$$

The map $C(W, \mathbb{T}) \rightarrow H^1(X, \mathbb{Z})$ is the composite $C(W, \mathbb{T}) \rightarrow H^1(W, \mathbb{Z}) \xrightarrow{p^*} H^1(X, \mathbb{Z})$. Its cokernel is $H^1(X, \mathbb{Z})/p^*(H^1(W, \mathbb{Z}))$ which is the image of $p_! : H^1(X, \mathbb{Z}) \rightarrow H^0(W, \mathbb{Z})$ by the Gysin

sequence. The image can only be 0 or \mathbb{Z} if X is connected. Using the isomorphism mentioned in the previous item of this lemma, we see that we need to find the connecting map $\text{im}(p_!) \rightarrow H_M^1(\mathbb{R}, C(X, \mathbb{T})_0) \simeq C(W, \mathbb{R})$. This map sends any class in $H^0(W, \mathbb{Z})$ to a constant \mathbb{Z} -valued function on W .

The above exact sequence now becomes

$$(3) \quad 0 \rightarrow \text{im}(p_!) \rightarrow C(W, \mathbb{R}) \rightarrow H_M^1(\mathbb{R}, C(X, \mathbb{T})) \rightarrow 0.$$

So $H_M^1(\mathbb{R}, C(X, \mathbb{T}))$ is isomorphic to the quotient of $C(W, \mathbb{R})$ by $\text{im}(p_!)$. It surjects onto the quotient of $C(W, \mathbb{R})$ by all of $H^0(W, \mathbb{Z})$ which is isomorphic to $C(W, \mathbb{T})_0$. \square

Lemma 2.1. *We have a commutative diagram with exact rows and columns*

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & H_M^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) & \xrightarrow{q} & C(W, \mathbb{T})_0 & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 0 & \longrightarrow & \ker(F) & \longrightarrow & \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) & \xrightarrow{F} & \text{Br}_{\mathbb{R}}(X) \longrightarrow 0 \\
 & & \eta \downarrow & & & & \\
 & & H^2(X, \mathbb{Z}) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

Proof. The vertical and horizontal short exact sequences above are of the form $0 \rightarrow B_i \rightarrow B_{i+1} \rightarrow B_{i+1}/B_i \rightarrow 0$ where the B_i are the groups in the filtration of $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(W)$ described in the unnumbered Theorem on page (153) of Ref. [14]. All we need to check is that $B_1 = H_M^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$. This will follow from the fact that $B_1 = H_M^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))/\text{im}(d'_2)$ and the fact that $\text{im}(d'_2) \subseteq H_M^3(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$. The last group vanishes by the result in Ref. [12], Thm. (3.1). The maps F, η, q were defined in the previous lemma. \square

We now make the following dictionary

- $y \in \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$, y not in $\ker(F) \Leftrightarrow$ Space X with $H \neq 0$. Here, $H = F(y)$.
- Element $y \in \ker(F) \subseteq \text{Br}_{\mathbb{R} \times \mathbb{Z}}$, y not in $H_M^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) \Leftrightarrow$ Space W with $H = 0$, $B \neq 0$. Here, $B = \eta(y)$.

- $y \in H_M^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) \subseteq \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \Leftrightarrow$ Space X with $H = 0, B = 0, A \neq 0$. We obtain an element of $C(W, \mathbb{T})_0$ which always corresponds to the zero class in $H^1(X, \mathbb{Z})$. However, this may instead be viewed as a collection of elements in $C(X, \mathbb{R})$ which are constant on the S^1 -orbits of X . Any two of these elements differ by an element of $C(X, \mathbb{Z}) \simeq H^0(X, \mathbb{Z})$.

This should be compared with the allowed gauge transformations of the B -field in string theory.

- $C_0(X, \mathcal{K})$ with the lift of the \mathbb{R} -action and the trivial \mathbb{Z} -action \Leftrightarrow Space X with $H = 0, B = 0, A = 0$.

Note that the last item above is exactly the C^* -dynamical system assigned to a space X with zero H -flux in Ref. [1]. If $y \in \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$, we have $[H] = F(y)$. Then $F^{-1}(y)$ is the coset $s(y) \circ \ker(F)$ and we note that we may change y by an element x of $\ker(F)$.

We had said earlier that \mathcal{A} should be viewed as a ‘topological approximation’ to the gerbe on spacetime. If we act on the gerbe on spacetime by a gerby gauge transformation, this corresponds to acting on \mathcal{A} by an automorphism. It is clear that shifting y by an element x of $\ker(F)$ corresponds to this automorphism. We may view this action as affecting the H -flux on spacetime by shifting the B -field, that is, if $H = dB$ we act by a gerbe automorphism to obtain $H = dB'$. As argued at the beginning of this section we are restricting ourselves to integral shifts of the B -field and so we only allow shifts by elements of $H^2(X, \mathbb{Z}) \xrightarrow{i} H^2(X, \mathbb{R})$, where i is the canonical inclusion. We note that $i \circ \eta(x)$ can only equal $[(B' - B)]$. That is, an element of the group $\ker(F)$ acts on the H -flux by shifting it by $i \circ \eta(x) = [(B - B')]$.

Similarly, if $y \in \ker(F)$, then $H = 0, B = \eta(y)$ and changing y by an element z of $B_1 = H_M^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$ doesn’t change B but might correspond to making a change in the A -field, the gauge field of the B -field.

Suppose we pick a principal S^1 -bundle E_p over W for each $p \in H^2(W, \mathbb{Z})$. Let $P_3(W) = \bigoplus_{p \in H^2(W, \mathbb{Z})} \text{Br}_{\mathbb{R}}(E_p)$. In Refs. [2, 13] the authors describe a set-valued functor P on the category of unbased CW -complexes. For any CW -complex W , $P(W)$ is the set of isomorphism classes of pairs of the form **(principal circle bundle, H -flux)** over W (see Ref. [2] for details). Ref. [2] show that there is a classifying space R_3 for the functor P and $P(W) = [W, R_3]$. Further, Ref. [2] shows that that T-duality is a natural transformation from P to itself giving a map we denote as $T : P \rightarrow P$. Since $\text{Br}_{\mathbb{R}}(E_p)$ is isomorphic to $H^3(E_p, \mathbb{Z})$, it is clear that an isomorphism class of a pair over W (in the sense of Ref. [2]) determines and is determined by an

element of P_3 . This extends to a natural isomorphism between P and P_3 because of the naturality of the cohomology groups. In particular, for any CW complex W , we have that $[W, R_3] = P_3(W)$.

Let $\mathcal{R}(W) = \bigoplus_{p \in H^2(W, \mathbb{Z})} \text{Br}_{\mathbb{R} \times \mathbb{Z}}(E_p)$. This too is a set-valued functor on the category of unbased CW-complexes. Then, we have the following theorem

Lemma 2.2. *There is a well-defined map $T : \mathcal{R}(W) \rightarrow \mathcal{R}(W)$ induced by the crossed product.*

Proof. We need the following well-known fact. Let \mathcal{A}, \mathcal{B} be C^* -algebras with G -action α, β respectively. Let $C_c(G, \mathcal{A})$ the α -twisted convolution algebra of \mathcal{A} -valued functions on G which are of compact support on G . Similarly, let $C_c(G, \mathcal{B})$ be the β -twisted convolution algebra of \mathcal{B} -valued functions on G which are of compact support on G . Give $C_c(G, \mathcal{A})$ and $C_c(G, \mathcal{B})$ the inductive limit topology⁹.

Theorem 2.2. *Suppose that (\mathcal{A}, G, α) and (\mathcal{B}, G, β) are dynamical systems and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is an equivariant homomorphism. Then, there is a homomorphism $\phi \rtimes \text{id} : \underset{\alpha}{\mathcal{A} \rtimes G} \rightarrow \underset{\beta}{\mathcal{B} \rtimes G}$ mapping $C_c(G, \mathcal{A})$ into $C_c(G, \mathcal{B})$ such that $\phi \rtimes \text{id}(f)(s) = \phi(f(s))$.*

From the proof of this theorem, it is clear that the extension $\phi \rtimes \text{id}$ is unique.

Here, $\mathcal{A} = \mathcal{B}$ and $G = \mathbb{R}$. Suppose $y \in \text{Br}_{\mathbb{R} \times \mathbb{Z}}(W)$, and we pick a representative $(\mathcal{A}, \alpha \times \phi)$ of y . We may define $T(y)$ to be the dynamical system $(\underset{\alpha}{\mathcal{A} \rtimes \mathbb{R}}, \alpha^\# \times \phi^\#)$ where $\phi^\#$ is the map induced on the crossed product by $\phi^\#(f)(t) = \phi \rtimes \text{id}(f)(t)$. It is clear that it is unique and commutes with the \mathbb{R} -action.

Changing the representative to a Morita equivalent one $(\mathcal{A}', \alpha' \times \phi')$ will not change the Morita equivalence class of the answer because, by the theorem cited in Ref. [14], Sec. (6.2), $\phi^{\# \prime}$ has the same Phillips-Raeburn obstruction as $\phi^\#$ and hence $\phi^{\# \prime}$ is exterior equivalent to $\phi^\#$. \square

Note that $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ has a large continuous part, namely the elements of $H_M^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$. If we quotient $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ by this continuous part (this is equivalent to considering $\text{im}(F_1)$), we may associate to X the group $H^2(X, \mathbb{Z}) \oplus H^3(X, \mathbb{Z})$. That is, given a CW-complex W we consider triples (**principal bundle, closed integral two-form, H -flux**) over W . In Section (6) below, we consider a set valued functor

⁹See Ref. [9], Corollary 2.48.

$P_{3,2}(W)$ which associates to each W the isomorphism classes of triples over W .

As we noted above if we fix a closed three-form H such that $[H]$ is the H -flux and a two-form field B such that $H = dB$ then any other two-form field B' with $H = dB'$ will be associated to a gauge transformation and also to a cohomology class $(B - B') \in H_{\text{de Rham}}^2(X)$ as $d(B - B') = 0$. Thus, in a triple $([p], b, H)$ the class b parametrizes gauge transformations. The class b should not be identified with the physical Kalb-Ramond field unless $H = 0$.

With this caveat in mind we will refer to the class b as ‘the B -field’ or ‘the B -class’ in what follows. It should be clear, however, that this is NOT, in fact, the physical Kalb-Ramond field, but a shift in the Kalb-Ramond field B that is, the characteristic class of a large gauge transformation of the H -flux.

By construction, we know that if we ‘forget’ the \mathbb{Z} -action on \mathcal{A} , the T-dual doesn’t change in either H -flux or topology. Hence, the map $F : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \rightarrow \text{Br}_{\mathbb{R}}(X)$ extends to a natural transformation between the functors \mathcal{R} and P_3 . Now, if we have two T-dual (in the sense above) C^* -dynamical systems $(\mathcal{A}, \mathbb{R} \times \mathbb{Z}, \alpha)$ and $(\mathcal{B}, \hat{\mathbb{R}} \times \mathbb{Z}, \beta)$ then the dynamical systems $(\mathcal{A}, \mathbb{R}, \alpha|_{\mathbb{R}})$ and $(\mathcal{B}, \hat{\mathbb{R}}, \beta|_{\hat{\mathbb{R}}})$ are dual. Hence, the following diagram commutes

$$\begin{array}{ccc} \mathcal{R}(W) & \xrightarrow{T} & \mathcal{R}(W) \\ F \downarrow & & F \downarrow \\ P_3(W) & \xrightarrow{T_3} & P_3(W). \end{array}$$

In the next section we attempt to determine a classifying space for ‘pairs’ (**principal bundle, Class in H^2**). If E_p is as above, such a pair will not specify an element in $\ker(F) \subset \text{Br}_{\mathbb{R} \times \mathbb{Z}}(E_p)$ uniquely, but only up to elements of B_1 . That is, for any unbased CW complex W , $[W, R_2] = \bigoplus_{p \in H^2(W, \mathbb{Z})} (\ker(F))^p / B_1^p$. The p -index on $\ker(F)$ and B_1 denote that they are subgroups of $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X_p)$.

As pointed out above, by considering $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X_p) / B_1$, we are led to consider ‘triples’ of the form (**principal bundle, Class in H^2 , Class in H^3**), here the class in H^3 is to be viewed as an H -flux. In the next section we argue that the assignment $W \rightarrow (\text{Triples over } W)$ is a set-valued functor (denoted $P_{3,2}(W)$) on the category of unbased CW complexes. There is a natural transformation π between these two functors obtained from the forgetful map $F_1 : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X_p) \rightarrow \text{Br}_{\mathbb{Z}}(X_p)$.

Given a C^* -dynamical system $(\mathcal{A}, \mathbb{R} \times \mathbb{Z}, \alpha)$ corresponding to a class $a \in \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X_p)$, adding an element of B_1 to a will not change the

Phillips-Raebrun invariant of $\alpha|_{\mathbb{Z}}$ or the H -flux $F(a)$. We showed in Ref. [12] that under T-duality the Phillips-Raebrun invariant of the \mathbb{Z} -action on $\mathcal{A} \rtimes_{\alpha|_{\mathbb{Z}}} \mathbb{R}$ associated to $T(a)$ only depends on the H -flux and the Phillips-Raebrun invariant of the dynamical system associated to a . It doesn't depend on the lift of these data to a $\mathbb{R} \times \mathbb{Z}$ -action on \mathcal{A} . Hence, we have a well-defined natural transformation $T_{3,2} : R_{3,2} \rightarrow R_{3,2}$ inducing a map $T_{3,2} : P_{3,2}(W) \rightarrow P_{3,2}(W)$ such that the following diagram commutes

$$(4) \quad \begin{array}{ccc} \mathcal{R}(W) & \xrightarrow{T_R} & \mathcal{R}(W) \\ \pi \downarrow & & \pi \downarrow \\ P_{3,2}(W) & \xrightarrow{T_{3,2}} & P_{3,2}(W). \end{array}$$

Also, $\pi : \mathcal{R}(W) \rightarrow P_{3,2}(W)$ is always surjective. Thus we may infer properties of T_R from those of $T_{3,2}$ but it should be clear that they are not the same. In this paper we mainly study $P_{3,2}$ and $T_{3,2}$. This is sufficient to answer the question we raised in Section (1).

In a recent thesis by Schneider [13] the author defines the collection of equivalence classes of C^* -dynamical systems (\mathcal{A}, G, α) whose spectrum is a principal G/N -bundle over W (denoted $\text{Dyn}^+(W)$). He then defines a T-duality map induced by the crossed product between equivalence classes of such dynamical systems. The resulting dynamical system is $(\mathcal{A} \rtimes_{\alpha^\#} \hat{G}, \hat{G}, \alpha^\#)$ and has spectrum a principal \hat{G}/N^\perp -bundle over W .

Thus the duality map in Ref. [13] would map systems of the form $(\mathcal{A}, \mathbb{R} \times \mathbb{Z}, \alpha)$ with spectrum a principal S^1 -bundle over W to those of the form $(\mathcal{A}^\#, \hat{\mathbb{R}} \times \mathbb{T}, \hat{\alpha})$ with a spectrum-fixing \mathbb{T} -action and spectrum a principal S^1 -bundle over W .

This map is not the same as the T-duality map we are considering here, since we map equivalence classes of C^* -dynamical systems of the form $(\mathcal{A}, \mathbb{R} \times \mathbb{Z}, \alpha)$ with spectrum a principal S^1 -bundle over W and a spectrum-fixing \mathbb{Z} -action on \mathcal{A} to those of the form $(\mathcal{A}^\#, \hat{\mathbb{R}} \times \mathbb{Z}, \hat{\alpha})$. Here $\mathcal{A}^\#$ has spectrum a principal S^1 -bundle over W but there is a *spectrum-fixing* \mathbb{Z} -action on $\mathcal{A}^\#$.

3. THE CLASSIFYING SPACE OF k -PAIRS

In this section, we use the method of Ref. [2] to determine the map T of Sec. (1).

Let \mathcal{SET} be the category of sets with functions as morphisms. Let \mathcal{C} be the category of unbased CW complexes with unbased homotopy

classes of continuous maps as morphisms. Let \mathcal{C}_0 be the category of CW complexes which are finite subcomplexes of some fixed countably infinite dimensional standard simplicial complex. $(\mathcal{C}, \mathcal{C}_0)$ is a homotopy category in the sense of Ref. [15] (see Thm. 2.5 in [15]).

Let W be a *fixed* CW complex. We define a k -pair over W to consist of a principal S^1 -bundle $p : E \rightarrow W$ together with a cohomology class $b \in H^k(W, \mathbb{Z})$. We denote a k -pair as $([p], b)$. (Here the space W is understood from the context as is the value of k .) Note that a ‘pair’ in the sense of Ref. [2] would be termed a 3-pair here.

Def 3.1. *We declare two k -pairs (same k) $([p], b)$ and $([q], b')$ over W equivalent if*

- *We are given two principal S^1 -bundles $p : E \rightarrow W$ and $q : E' \rightarrow W$ such that*

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ p \downarrow & & q \downarrow \\ W & \xrightarrow{id} & W \end{array}$$

commutes.

- *We also require that $b' = \phi^*(b)$.*

It is clear that the collection of equivalence classes of k -pairs over a fixed space W (denoted $P_k(W)$) is a *set*. For all W , we have a distinguished pair consisting of the trivial S^1 -bundle over W with the zero class in $H^k(W, \mathbb{Z})$. Thus, $P_k(W)$ is actually a pointed set.

Def 3.2. *Let W and Y be two CW-complexes and let $f : W \rightarrow Y$ be a continuous map. Let $([p], b) \in P_k(Y)$ be represented by a principal S^1 -bundle $p : E \rightarrow Y$ and a class $b \in H^k(Y, \mathbb{Z})$. We define the pullback of $([p], b)$ via f , denoted $f^*([p], b)$, to be the following data*

- *The unique principal S^1 -bundle $f^*p : f^*E \rightarrow W$ such that the following diagram commutes*

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{\phi}_f} & E \\ f^*p \downarrow & & p \downarrow \\ W & \xrightarrow{f} & Y. \end{array}$$

- *The cohomology class $\phi_f^*(b)$ in $H^k(f^*E, \mathbb{Z})$.*

That is, we define $f^([p], b) = (f^*[p], \phi_f^*(b))$.*

Lemma 3.1. *Let $f_0, f_1 : W \rightarrow Y$ be freely homotopic. For any pair $([p], b) \in P_2(Y)$, $f_0^*([p], b)$ is equivalent to $f_1^*([p], b)$.*

Proof. Let $p : E \rightarrow Y$ be a principal S^1 -bundle. We have pullback squares for $i = 0, 1$

$$\begin{array}{ccc} f_i^* E & \xrightarrow{\tilde{\phi}_i} & E \\ f_i^* p \downarrow & & p \downarrow \\ W & \xrightarrow{f_i} & Y. \end{array}$$

Then, by Ref. [16], Cor. (1.8), the pullback bundles $f_0^* p : f_0^* E \rightarrow W$ and $f_1^* : f_1^* E \rightarrow W$ are isomorphic. Further, by the same lemma, this isomorphism is implemented by a map $\psi : f_0^* E \rightarrow f_1^* E$. This map induces isomorphisms on the cohomology groups such that $\psi \circ f_0^* = f_1^*$. As a result, by the above definition, $f_0^*([p], b) = f_1^*([p], b)$. \square

Hence, $P_k(W)$ is a pointed set depending only on the homotopy type of W . Given a map $f : W \rightarrow Y$, define $P_k(f) : P_k(Y) \rightarrow P_k(W)$ to be the map induced by pullback of pairs. It is clear that $P_k(1) = \text{Id}$. (This is just the condition that two pairs be equivalent). Hence, P_k extends to a functor (also denoted P_k) $P_k : \mathcal{C} \rightarrow \mathcal{SET}$.

Theorem 3.1. *For every k , the functor P_k above satisfies the conditions of the Brown Representability Theorem. Hence, for every k , there exists a classifying space R_k for P_k .*

Proof. There are two conditions we need to prove.

- (1) Consider an arbitrary family $\{W_\mu\}, \mu \in I$ of objects in \mathcal{C} . Let $Y = \bigsqcup_{\mu \in I} W_\mu$. Let $h_\mu : W_\mu \rightarrow \bigsqcup_{\mu \in I} W_\mu$ be the inclusion maps.

We have a pullback square (for every $\mu \in I$)

$$\begin{array}{ccc} h_\mu^* E & \xrightarrow{\tilde{h}_\mu} & E \\ p_\mu \downarrow & & p \downarrow \\ W_\mu & \xrightarrow{h_\mu} & Y. \end{array}$$

Here $p_\mu = h_\mu^* p$. Since $H^2(W, \mathbb{Z}) \simeq \prod_{\mu \in I} H^2(W_\mu, \mathbb{Z})$, we have that $[p] = ([p_\mu]), \mu \in I$.

Let $h_\mu^* E = E_\mu$, then, we also have that $E = \bigsqcup_{\mu \in I} E_\mu$ and $H^k(E, \mathbb{Z}) \simeq \prod_{\mu \in I} H^k(E_\mu, \mathbb{Z})$. Hence, every class $b \in H^k(E, \mathbb{Z})$ may be written as $(b_\mu), \mu \in I$ with $b_\mu = \tilde{h}_\mu^*(b)$. Hence, we have an isomorphism

$$\Pi_\mu P(h_\mu) : P\left(\bigsqcup_{\mu} W_\mu\right) \approx \Pi_\mu P(W_\mu).$$

- (2) Suppose we are given CW complexes A, W_1, W_2 and continuous maps $f_i : A \rightarrow W_i, g_i : W_i \rightarrow Z, i = 1, 2$ such that

$$\begin{array}{ccc} A & \xrightarrow{f_1} & W_1 \\ f_2 \downarrow & & \downarrow g_1 \\ W_2 & \xrightarrow{g_2} & Z \end{array}$$

commutes up to homotopy and is a pushout square in \mathcal{C} . We may take f_i to be inclusions into the W_i and Z the result of gluing W_1 to W_2 along A . Suppose $u_i \in P(W_i)$ satisfy $P(f_1)u_1 = P(f_2)u_2$. For $i = 1, 2$, let u_i correspond to the pair $([p_i], b_i)$ over W_i , where $p_i : E_i \rightarrow W_i$ are principal S^1 -bundles. Then, since $P(f_1)u_1 = P(f_2)u_2$, $f_1^*E_i \simeq f_2^*E_2$. This implies that the restrictions of $f_i^*E_i$ to A are the same. Hence, these two bundles may be glued into a unique bundle $p : E \rightarrow Z$. Note that $G_i = f_i^*E_i \subset E, i = 1, 2$ and $G_1 \cup G_2 = E$. We have a pullback square

$$\begin{array}{ccc} G_i & \xrightarrow{\tilde{g}_i} & E \\ p_i \downarrow & & \downarrow p \\ W_i & \xrightarrow{g_i} & Z \end{array}$$

By the Mayer-Vietoris theorem, we have

$$H^k(E, \mathbb{Z}) \rightarrow H^k(G_1, \mathbb{Z}) \oplus H^k(G_2, \mathbb{Z}) \rightarrow H^k(G_1 \cap G_2, \mathbb{Z})$$

Now $f_1^*(b_1) = f_2^*(b_2)$ and so the image of (b_1, b_2) via the second map above is zero. Hence, by exactness, there is an element in $c \in H^k(E, \mathbb{Z})$ such that $\tilde{g}_i(c) = b_i, i = 1, 2$. Thus, we define an element $v \in P(Z)$ by $v = ([p], c)$. It is clear that $P(g_i)v = u_i, i = 1, 2$.

As a result, for every k , there is a CW complex R_k such that isomorphism classes of k -pairs over a space W correspond to *unbased* homotopy classes of maps from $W \rightarrow R_k$. \square

Similarly we may define a k_1, k_2, \dots, k_n -tuple ($k_i \in \mathbb{N}$) over a space W to consist¹⁰ of a principal S^1 -bundle $p : E \rightarrow W$ together with cohomology classes b_1, \dots, b_n such that $b_i \in H^{k_i}(E, \mathbb{Z})$. Exactly as above we may define the notion of equivalent tuples and a set valued functor $P_{k_1, k_2, \dots, k_n}(W)$. As above, such a functor is representable and has a representation space denoted R_{k_1, k_2, \dots, k_n} . Note that there are natural

¹⁰Obviously the ordering of the k_i is irrelevant.

maps $R_{k_1, \dots, k_n} \rightarrow K(\mathbb{Z}, 2)$ and $R_{k_1, \dots, k_n} \rightarrow K(\mathbb{Z}, k_i - 1)$ given by sending $([p], b_1, \dots, b_i) \rightarrow [p]$ and $([p], b_1, \dots, b_i) \rightarrow p_i(b_i)$ respectively. In addition, the canonical S^1 -bundle U over $K(\mathbb{Z}, 2)$ defines a unique pair $(U, 0)$ over R_k for every k and hence the natural map $R_k \rightarrow K(\mathbb{Z}, 2)$ is naturally split. This implies that the cohomology ring of R_k contains $\mathbb{Z}[\alpha]$, $\alpha \in H^2(R_k, \mathbb{Z})$.

Given a 3, 2-tuple $([p], b, H)$ over W , we obtain a unique 2-pair $([p], b)$ and 3-pair $([p], H)$ over W . Similarly, given a 2-pair $([p], b)$ and a 3-pair $([p], H)$ (*same* $[p]$) over W , we obtain a unique triple $([p], b, H)$ over W . Thus, $R_{3,2}$ is a fiber product $R_3 \times_{K(\mathbb{Z}, 2)} R_2$. Similarly $R_{k_1, \dots, k_n} \simeq R_{k_1} \times_{K(\mathbb{Z}, 2)} \dots \times_{K(\mathbb{Z}, 2)} R_{k_n}$. There are natural fibrations $q_{k_i} : R_{k_1, \dots, k_n} \rightarrow R_{k_i}$.

For the sake of completeness we note the following

Lemma 3.2. *Suppose W is k -connected, $k \geq 2$. Then, $P_{k+1}(W) \simeq H^{k+1}(W, \mathbb{Z})$.*

Proof. Since W is at least 2-connected, all principal S^1 -bundles on W are trivial. Further, $H^{k+1}(W \times S^1, \mathbb{Z}) \simeq H^{k+1}(W, \mathbb{Z})$ and hence the result. \square

We will only consider 2-tuples, 3-tuples and 3, 2-tuples in this paper. We will also consider $P_2, P_3, P_{3,2}$ and the corresponding classifying spaces $R_2, R_3, R_{3,2}$.

Bunke et al. [2] have considered the case $k = 3$. We denote their classifying space R_3 here. For the remainder of this section and the next we work with $k = 2$. We abbreviate 2-pair to ‘pair’.

A priori, R_2 is an unbased CW complex. We now arbitrarily pick a basepoint r_0 in R_2 .

Lemma 3.3. *Let W be any CW complex. Pick a basepoint $x_0 \in W$. Any unbased map $f : W \rightarrow R_2$ may be freely homotoped to a based map $g : (W, x_0) \rightarrow (R_2, r_0)$.*

Proof. Suppose W was any CW complex, and $f : W \rightarrow R_2$ an unbased map. By the Lemma that follows, we know that R_2 is a fibration of a connected space over a connected base space. Hence R_2 is connected. Pick a basepoint $x_0 \in W$. Pick a path $q : I \rightarrow R_2$ connecting $f(x_0)$ to r_0 . Extend the data f, q to a free homotopy $H : W \times I \rightarrow R_2$. Then, $g = H(1, \cdot) : W \rightarrow R_2$ is map such that $g(x_0) = r_0$. The map g classifies the same pair that f does, since R_2 is an *unbased classifying space*. \square

Lemma 3.4. *There is a fibration $K(\mathbb{Z}, 2) \rightarrow R_2 \rightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$.*

Proof. Given a pair $([p], b)$ over W , we obtain two natural cohomology classes $[p] \in H^2(W, \mathbb{Z})$ and $p_!(b) \in H^1(W, \mathbb{Z})$. As a result, there is a natural map $\phi \times \psi : R_2 \rightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$. Given a pair $([p], b)$ over any space W , classified by $f : W \rightarrow R_2$, the map $f \mapsto \phi \circ f$ corresponds to the map $([p], b) \mapsto [p]$. Similarly, $f \mapsto \psi \circ f$ corresponds to the map $([p], b) \mapsto p_!(b)$. We pick a basepoint in $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$ such that $\phi \times \psi$ is a based map. Suppose $f : W \rightarrow R_2$ classified a pair $([p], b)$ over W . Pick a basepoint $x_0 \in W$. By Lemma 3.3 above, f may be freely homotoped to a based map $g : (W, x_0) \rightarrow (R_2, r_0)$. Suppose g was in the homotopy fiber of $\phi \times \psi$. Then, we would obtain a pair $([p], b)$ over W such that $p_!(b) = 0$, $[p] = 0$. This would correspond to the trivial bundle $W \times S^1 \rightarrow W$ equipped with the cohomology class $1 \times a$, $a \in H^2(W, \mathbb{Z})$. Hence we would get a natural based map $W \rightarrow K(\mathbb{Z}, 2)$. Conversely, given a class a in $H^2(W, \mathbb{Z})$, we could obtain a pair $(0, 1 \times a)$ over W which would have $[p] = 0$ and $p_!(1 \times a) = 0$. By Lemma 3.3, this pair would be classified by a based map $g : (W, x_0) \rightarrow (R_2, r_0)$. Obviously, $(\phi \times \psi) \circ g$ would be nullhomotopic.

Thus, the homotopy fiber of $\phi \times \psi$ is $K(\mathbb{Z}, 2)$. \square

Lemma 3.5. *The homotopy groups of R_2 are as follows*

- $\pi_1(R_2) = \mathbb{Z}$,
- $\pi_2(R_2) \simeq \mathbb{Z}^2$,
- $\pi_i(R_2) = 0, i > 2$.

Proof. We had picked a basepoint for R_2 . Hence, we may calculate $\pi_i(R_2)$ from the long exact sequence of the fibration in Lemma 3.4. We find that the nonzero part of the sequence is

$$0 \rightarrow \mathbb{Z} \rightarrow \pi_2(R_2) \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \pi_1(R_2) \rightarrow \mathbb{Z} \rightarrow 0.$$

Thus, $\pi_1(R_2) = \mathbb{Z}$, $\pi_2(R_2) \simeq \mathbb{Z}^2$, and $\pi_i(R_2) = 0, i > 2$. \square

We may characterize R_2 as follows

Lemma 3.6. *Let $c : K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1) \rightarrow K(\mathbb{Z}, 3)$ be the based map which induces the cup product. Then R_2 is the homotopy fiber of c .*

Proof. Let $f : W \rightarrow R_2$ be a map inducing the pair $([p], b)$ over W . Fixing a basepoint $x_0 \in W$, we may replace f by a based map $g : (W, x_0) \rightarrow (R_2, r_0)$ by Lemma 3.3. It is clear that we may take $\phi \times \psi$ to be based. Then we have a principal S^1 -bundle $p : E \rightarrow W$. By the Gysin sequence of this bundle we have that $[p] \cup p_!(b) = 0$. This implies that $c \circ (\phi \times \psi) \circ f$ is nullhomotopic via a based homotopy, since c is exactly the based map which gives the cup product.

Conversely, suppose we are given a based map $f : W \rightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$ such that $c \circ f$ is nullhomotopic. Then, this corresponds to a

class $a \in H^1(W, \mathbb{Z})$ and a class $[p] \in H^2(W, \mathbb{Z})$ such that $[p] \cup a = 0$. Pick a principal S^1 -bundle $p : E \rightarrow W$ with characteristic class $[p]$. From the Gysin sequence of this bundle we see that $[p] \cup a = 0$ implies that $a = p_!(b)$ for some $b \in H^2(E, \mathbb{Z})$. Thus, we obtain a pair $([p], b)$ over W and hence an unbased map $g : W \rightarrow R_2$. By the above argument, we may replace it with a based map h classifying the *same* pair over W . Obviously, $(\phi \times \psi) \circ h = f$ as a based map.

Hence, R_2 is the homotopy fiber of the based map c in the category of based CW complexes with basepoint preserving homotopy classes of maps between them. There is a forgetful functor from this category to the category \mathcal{C} . We take the image of the homotopy fiber of c via this functor. This determines R_2 up to homotopy equivalence in \mathcal{C} . \square

Since $\pi_1(R_2) \neq 0$, the choice of basepoints might be important. Indeed, we have the following

Lemma 3.7. *The space R_2 is not simple.*

Proof. Suppose R_2 was simple: Then, from Postnikov theory, we see that R_2 would be homotopy equivalent to the product $K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2)$ via a based homotopy. Given $f : W \rightarrow R_2$ by Lemma 3.3, we could obtain a based map $g : W \rightarrow R_2$ classifying the same pair over W as f . Hence we would obtain based maps $W \rightarrow K(\mathbb{Z}, 2)$, $W \rightarrow K(\mathbb{Z}, 2)$ and $W \rightarrow K(\mathbb{Z}, 1)$. The pair would then be *determined* by classes $[p], a \in H^2(W, \mathbb{Z})$ and $p_!(b) \in H^1(W, \mathbb{Z})$. Here $[p]$ would be the characteristic class of a principal S^1 -bundle $p : E \rightarrow W$. This would imply in turn that b would be determined by $p_!(b)$ and a and hence that the Gysin sequence for $p : E \rightarrow W$ would split at degree two for *any* principal S^1 -bundle E over W . Since W, E were arbitrary, this is obviously impossible. \square

Lemma 3.8. *The cohomology of R_2 up to degree 3 is*

- $H^0(R_2, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^1(R_2, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^2(R_2, \mathbb{Z}) \simeq \mathbb{Z}$.
- $H^3(R_2, \mathbb{Z}) \simeq \mathbb{Z}$.

Proof. Consider the fibration $K(\mathbb{Z}, 2) \rightarrow R_2 \rightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$. We have that $H^*(K(\mathbb{Z}, 2), \mathbb{Z}) \simeq \mathbb{Z}[a]$ where a is a generator of $H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \simeq \mathbb{Z}$. This ring has no automorphisms apart from $a \rightarrow -a$. Since the fibration is oriented, the generator of $\pi_1(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1))$ acts trivially on the cohomology of $K(\mathbb{Z}, 2)$. As a result, we may use the Serre spectral sequence using cohomology with untwisted coefficients to calculate $H^*(R_2, \mathbb{Z})$.

We note that the above fibration is pulled back from the path-loop fibration over $K(\mathbb{Z}, 3)$ via the map $c : K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1) \rightarrow K(\mathbb{Z}, 3)$ inducing the cup product. Suppose μ was a generator of $H^1(K(\mathbb{Z}, 1), \mathbb{Z})$ and that λ was a generator of $H^2(K(\mathbb{Z}, 2), \mathbb{Z})$. Let $\tilde{\mu} = (\phi \times \psi)^*(\mu)$, and $\tilde{\lambda} = (\phi \times \psi)^*(\lambda)$. Then we have that $\tilde{\mu} \cup \tilde{\lambda} = 0$. This shows that the transgression $E_2^{2,0} \rightarrow E_2^{0,3}$ must be a map $k : H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \simeq \mathbb{Z} \rightarrow H^2(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1), \mathbb{Z}) \simeq \mathbb{Z}$ sending $a \rightarrow a(\mu \cup \lambda)$.

From the spectral sequence table, we see that $H^0(R_2, \mathbb{Z}) \simeq \mathbb{Z}$, $H^1(R_2, \mathbb{Z}) \simeq \mathbb{Z}$, $H^2(R_2, \mathbb{Z}) \simeq \mathbb{Z}$ and $H^3(R_2, \mathbb{Z}) \simeq \mathbb{Z}$. \square

Note that the canonical bundle over $K(\mathbb{Z}, 2)$ gives rise to a unique pair $(1, 0)$ over $K(\mathbb{Z}, 2)$. This is classified by a map $K(\mathbb{Z}, 2) \rightarrow R_2$ which is a section of the natural map $R_2 \rightarrow K(\mathbb{Z}, 2)$. Hence, the cohomology ring of R_2 contains $\mathbb{Z}[b]$, $b \in H^2(R_2, \mathbb{Z})$.

This ring has generators a, b, c in degrees 1, 2 and 3 respectively such that

$$(5) \quad a.b = 0, b^n \neq 0 \text{ for any } n.$$

We now determine the action of $\pi_1(R_2)$ on $\pi_2(R_2)$.

Theorem 3.2. *The action of the generator S of $\pi_1(R_2) \simeq \mathbb{Z}$ on $\pi_2(R_2)$ is given by*

$$S(a, b) = (a + b, b).$$

Proof. From the long exact sequence of homotopy groups of the fibration in Lemma 3.4, we see that we have a sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z} \rightarrow 0$. Now, $\pi_1(R_2)$ acts on each term of this sequence by \mathbb{Z} -module automorphisms with the trivial action on the first and last \mathbb{Z} factors and by an action θ on the middle factor.

This implies that θ may be taken to be the homomorphism induced by the matrix

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

We claim that

$$\theta \simeq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We have a fibration $K(\mathbb{Z}, 2) \rightarrow R_2 \rightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$. We could view this as a fibration over the $K(\mathbb{Z}, 1) \simeq S^1$ factor $K(\mathbb{Z}^2, 2) \rightarrow R_2 \rightarrow S^1$. From the long exact sequence of a fibration, it is clear that $K(\mathbb{Z}^2, 2)$ is the universal cover \tilde{R}_2 of R_2 . Now $\pi_1(S^1)$ acts on \tilde{R}_2 by deck transformations. Hence, using the Cartan-Leray spectral sequence (See Ref. [17] Ch. XVI Sec. 9 for details), we have a spectral sequence with

$E_2^{p,q} = H^p(\mathbb{Z} = \pi_1(S^1), H^q(K(\mathbb{Z}^2, 2), \mathbb{Z})) \Rightarrow H^*(R_2, \mathbb{Z})$ (here $H^*(\mathbb{Z}, M)$ denotes the group cohomology of \mathbb{Z} with coefficients in a module M). This sequence collapses at the E_2 term itself, since $E_2^{p,q} \simeq 0$ for $p \geq 2$. Thus $\mathbb{Z} \simeq H^2(R_2, \mathbb{Z}) \simeq H^0(\mathbb{Z}, H^2(K(\mathbb{Z}^2, 2), \mathbb{Z})) \simeq H^0(\mathbb{Z}, \mathbb{Z}^2)$, and so the fixed points of θ on \mathbb{Z}^2 are $\mathbb{Z} \neq \mathbb{Z}^2$. Hence the action θ is *not* trivial and R_2 is *not* simple.

Now, $\mathbb{Z} \simeq H^3(R_2, \mathbb{Z}) \simeq H^1(\mathbb{Z}, H^2(K(\mathbb{Z}^2, 2), \mathbb{Z})) \simeq H^1(\mathbb{Z}, \mathbb{Z}^2)$. If \mathbb{Z} acts on \mathbb{Z}^2 with an action θ , $H^*(\mathbb{Z}, \mathbb{Z}^2)$ is the cohomology of the complex $\mathbb{Z}^2 \xrightarrow{\theta-1} \mathbb{Z}^2$. Hence, $\mathbb{Z}^2/(\theta-1)\mathbb{Z}^2 \simeq \mathbb{Z}$ here, and using the above form for θ ,

$$\theta \simeq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

□

Note that since R_2 was defined in the unbased category, for any CW-complex W , pairs over W are classified by unbased maps from W to R_2 . Since the space of unbased maps from W to R_2 is the quotient of the space of based maps from W to R_2 by the action of $\pi_1(R_2)$, we see that the non-trivial action of $\pi_1(R_2)$ does not affect our results. We simply have to be careful to use unbased maps in all our constructions. We can see an example of this when we try to determine all the pairs over S^2 . If pairs were classified by based maps, then $P_2(S^2)$ would be a group. However, we have the following

Lemma 3.9. *$P_2(S^2)$ is not a group.*

Proof. For any CW complex W , we have a natural map $\phi : P_2(W) \rightarrow H^2(W, \mathbb{Z})$ which sends a pair $([p], b)$ over W to $[p]$. Now, for every $a \in H^2(S^2, \mathbb{Z}) \simeq \mathbb{Z}$, we claim that the set $\phi^{-1}(a)$ has cardinality $|a|$. To see this, it is enough to note that if $D_p \rightarrow S^2$ is a principal S^1 -bundle of Chern class $[p]$, then $H^2(D_p, \mathbb{Z}) \simeq \mathbb{Z}_p$. This implies that $P_2(S^2) \rightarrow H^2(S^2, \mathbb{Z})$ is not a group homomorphism, and hence that $P_2(S^2)$ is not a group. □

In fact, $P_2(S^2)$ is the quotient of the group $\pi_2(R_2) \simeq \mathbb{Z}^2$ by the action of $\pi_1(R_2)$ calculated in Thm. 3.2 above.

Lemma 3.10. *Consider the principal S^1 -bundle $p : E_2 \rightarrow R_2$ whose characteristic class is b the generator of $H^2(R_2, \mathbb{Z})$. Its cohomology groups are*

- $H^0(E_2, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^1(E_2, \mathbb{Z}) \simeq \mathbb{Z}\bar{a}$,
- $H^2(E_2, \mathbb{Z}) \simeq \mathbb{Z}w$,
- $H^3(E_2, \mathbb{Z}) \simeq \mathbb{Z}\bar{c}$.

where \bar{a}, \bar{c} are the images of the generators a, c of $H^i(R_2, \mathbb{Z})$, $i = 1, 3$ in $H^*(E_2, \mathbb{Z})$. In addition $p_!(w) = a$, $\bar{a} = p^*(a)$, $\bar{c} = p^*(c)$.

Proof. The Serre Spectral Sequence table is

$$(6) \quad \begin{array}{c|ccc} \mathbb{Z}z & \mathbb{Z}az & \mathbb{Z}bz & \mathbb{Z}cz \\ \hline \mathbb{Z} & \mathbb{Z}a & \mathbb{Z}b & \mathbb{Z}c \end{array}$$

The transgression map $d^2 : E_2^{0,1} \rightarrow E_2^{2,0}$ has to be $dz = b$. Hence using the ring structure, the fact that $ab = 0$ in $H^*(R_2, \mathbb{Z})$, and the fact that b generates a copy of $\mathbb{Z}[b]$ inside $H^*(E_2, \mathbb{Z})$ gives the cohomology groups shown. In addition, note that w is the image of $a.z$ in $H^2(E_2, \mathbb{Z})$.

Then, an inspection of the Gysin sequence for E_2 shows that the elements \bar{a}, \bar{c} must be images of a, c under p^* . \square

This bundle acts as a universal bundle for pairs: Given a principal S^1 -bundle $p : D \rightarrow W$ and $b \in H^2(D, \mathbb{Z})$ we have a classifying map $f : W \rightarrow R_2$. This bundle pulls back along f to give the bundle $p : D \rightarrow W$ while the generator of $H^2(E_2, \mathbb{Z})$ pulls back to the b -class.

We hope to study the map T of Section (1) using the classifying space R_2 studied above.

4. T-DUALITY FOR AUTOMORPHISMS IS NOT INVOLUTIVE

By the proof of Thm. (3.1) in Ref. [12], we know that the T-dual of an automorphism even with H -flux is always unique. However, T-duality for automorphisms is not involutive: If we perform two successive T-dualities we may not get the automorphism we started with. For example, if $W = S^2$ with 1 unit of H -flux on $S^2 \times S^1$, the T-dual is S^3 with no H -flux. Since $H^2(S^2 \times S^1, \mathbb{Z}) \simeq \mathbb{Z}$, but $H^2(S^3, \mathbb{Z}) \simeq 0$, every locally unitary (but not necessarily unitary) automorphism of $CT(S^2 \times S^1, 1)$ dualizes to a unitary automorphism of $C(S^3, \mathcal{K})$. From the proof of Thm. (3.1) of Ref. [12], it is clear that the T-dual of a unitary automorphism is unitary. Hence, taking one more T-dual gives a unitary automorphism of $CT(S^2 \times S^1, 1)$.

At the level of triples $([p], b, H)$ we conjecture that the T-duality map of Section (1) should have the following properties:

Lemma 4.1. *Let W be connected and simply connected. Let $p : E \rightarrow W$ be a principal S^1 -bundle with H -flux H and $b \in H^2(E, \mathbb{Z})$. Let $q : E^\# \rightarrow W$ be the T-dual principal S^1 -bundle with H -flux $H^\#$ and $b^\# \in H^2(E^\#, \mathbb{Z})$ where $b^\# = T(b)$. Then, for all $b \in H^2(E, \mathbb{Z})$, $\forall l, m \in \mathbb{Z}$, the Gysin sequence induces a bijection between the cosets*

$$\{b + lp^*p_!(H)\} \text{ and } \{b^\# + mq^*q_!(H^\#)\}.$$

Proof: The Gysin sequence of $p : E \rightarrow W$ is $\mathbb{Z} \xrightarrow{[p]} H^2(W, \mathbb{Z}) \xrightarrow{p^*} H^2(E, \mathbb{Z}) \rightarrow \dots$. The kernel of p^* is the subgroup¹¹ $\langle [p] \rangle$. Similarly, the kernel of q^* is the subgroup $\langle [q] \rangle$. Let $G = \langle [p], [q] \rangle = \langle p_!(H), q_!(H^\#) \rangle$. Since $H^1(W, \mathbb{Z}) = 0$, p^*, q^* are surjective by the Gysin sequence. Note that $p^*G \simeq \langle p^*p_!(H) \rangle$ and $q^*G \simeq \langle q^*q_!(H^\#) \rangle$. Thus, we have isomorphisms $H^2(W, \mathbb{Z})/G \simeq H^2(E, \mathbb{Z})/\langle p^*p_!(H) \rangle \simeq H^2(E^\#, \mathbb{Z})/\langle q^*q_!(H^\#) \rangle$.

□

Note that in the special case $E^\# = W \times S^1$, $H^\# = [p] \times z$, where $p : E \rightarrow W$ is a principal S^1 -bundle, $H = 0$, the above theorem states that there is a natural map between the coset $\{a \times 1 + l[p] \times 1\}$ and $b^\#$. Here, b is always of the form $a \times 1$, $a \in H^2(W, \mathbb{Z})$, and $b^\# = p^*(a)$ by the Gysin sequence.

We conjecture that each coset is precisely the collection of b -fields with the same T -dual (even when X is **not** simply connected). As support for this, note the following: Suppose $E^\# = W \times S^1$, with k units of H -flux. Let $q = \pi : W \times S^1 \rightarrow W$ be the projection map. Then, if $H^1(W, \mathbb{Z}) \neq 0$, the above theorem would not be expected to hold: For one thing, $\text{im}(\pi^*)$ would not be all of $H^2(W \times S^1, \mathbb{Z})$. It is strange then, that $H^2(E, \mathbb{Z})/\langle p^*p_!(H) \rangle$ is isomorphic to $H^2(E^\#, \mathbb{Z})/\langle q^*q_!(H^\#) \rangle$ in all the following cases in many of which $H^1(W, \mathbb{Z}) \neq 0$ (I use Ref. [4] for the examples):

- (1) $W = T^2$: For $E^\# = W \times S^1$, $H^0(E^\#, \mathbb{Z}) = \mathbb{Z}$, $H^1 = \mathbb{Z}^3$, $H^2 = \mathbb{Z}^3$, $H^3 = \mathbb{Z}$. The H -flux is a class $[p] \times z \in H^2(T^2) \otimes H^1(S^1) \simeq H^3(T^2 \times S^1) \simeq \mathbb{Z}$. The T -dual is the nilmanifold $p : N \rightarrow T^2$ whose cohomology is $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^2$, $H^2 = \mathbb{Z}^2 \oplus \mathbb{Z}_p$, $H^3 = \mathbb{Z}$. It is clear that $\mathbb{Z}^3/p\mathbb{Z} \simeq \mathbb{Z}^2 \oplus \mathbb{Z}_p$.
- (2) $W = M$, an orientable surface of genus $g > 1$: The cohomology of $W \times S^1$ is $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^{2g+1}$, $H^2 = \mathbb{Z}^{2g+1}$, $H^3 = \mathbb{Z}$. The H -flux is a class $j \times z \in H^3 \simeq H^2(M) \otimes H^1(S^1) \simeq \mathbb{Z}$. The T -dual is a bundle $j : E \rightarrow M$ with $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^{2g}$, $H^2 = \mathbb{Z}^{2g} \oplus \mathbb{Z}_j$, $H^3 = \mathbb{Z}$. Here, $\mathbb{Z}^{2g+1}/j\mathbb{Z} \simeq \mathbb{Z}^{2g} \oplus \mathbb{Z}_j$.
- (3) $W = \mathbb{RP}^2$: The cohomology of $W \times S^1$ is $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}$, $H^2 = \mathbb{Z}_2 \simeq H^2(W) \otimes H^0(S^1)$, $H^3 \simeq H^2(W) \otimes H^1(S^1) \simeq \mathbb{Z}_2$. The H -flux is the class $1 \times z \in H^2(W) \otimes H^1(S^1)$. The T -dual is a bundle $k : E \rightarrow \mathbb{RP}^2$ with $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}$, $H^2 = 0$, $H^3 = \mathbb{Z}_2$. Once again, $\mathbb{Z}_2/1\mathbb{Z}_2 \simeq 0$.
- (4) $W = \mathbb{RP}^3$: The cohomology of $W \times S^1$ is $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}$, $H^2 = \mathbb{Z}_2$, $H^3 = \mathbb{Z} \oplus \mathbb{Z}_2$, $H^4 = \mathbb{Z}$. Then, $H^3 \simeq H^2(W) \otimes H^1(S^1) \oplus$

¹¹ $\langle a_1, a_2, \dots, a_n \rangle$ denotes the subgroup generated by a_1, a_2, \dots, a_n . The ambient group is understood from context.

$H^3(W) \otimes H^0(S^1)$. The H -flux is $1 \times z + k \times 1$. Now, $\pi^* \pi_1(H) = 1 \times 1$. The T-dual is $q : S^1 \times S^3 \rightarrow \mathbb{RP}^3$ with cohomology $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^3 = \mathbb{Z}, H^4 = \mathbb{Z}$. The T-dual has no B -class and H -flux $k \in \mathbb{Z}$.

- (5) $W = \mathbb{RP}^{2m}(m > 1)$: The cohomology of $W \times S^1$ is $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^q = \mathbb{Z}_2, q = 2, \dots, m-1, H^{2m} = \mathbb{Z}, H^{2m+1} = \mathbb{Z}_2$. The H -flux is the class $1 \times z \in H^2(\mathbb{RP}^{2m}) \otimes H^1(S^1)$. The T-dual is a bundle $q : E \rightarrow \mathbb{RP}^{2m}$ with cohomology $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^{2m+1} = \mathbb{Z}_2$. Here, $\mathbb{Z}_2/1\mathbb{Z}_2 \simeq 0$.
- (6) $W = \mathbb{RP}^{2m+1}(m > 1)$: The cohomology of $W \times S^1$ is $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^q = \mathbb{Z}_2, q = 2, \dots, m-1, H^{2m+1} = \mathbb{Z} \oplus \mathbb{Z}_2, H^{2m+2} = \mathbb{Z}$. Note $H^3 = \mathbb{Z}_2$; we have that $H^3 \simeq H^2(\mathbb{RP}^{2m+1}) \otimes H^1(S^1)$. The H -flux is the class $1 \times z \in H^3 \simeq \mathbb{Z}_2$. The T-dual is a principal bundle $q : S^1 \times S^{2m+1} \rightarrow \mathbb{RP}^{2m+1}$ with cohomology $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^{2m+1} = \mathbb{Z}, H^{2m+2} = \mathbb{Z}$. The T-dual has no second cohomology as expected.
- (7) $W = \mathbb{CP}^2$: The cohomology of $W \times S^1$ is $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^2 = \mathbb{Z}, H^3 = \mathbb{Z}, H^4 = \mathbb{Z}, H^5 = \mathbb{Z}$. We have $H^3 \simeq H^2(W) \oplus H^1(S^1) \simeq \mathbb{Z}$. The H -flux is the class $j \times z \in H^3$. The T-dual is the Lens space $L(2, j) \rightarrow \mathbb{CP}^2$ if $j \neq 0$. It has cohomology $H^0 = \mathbb{Z}, H^2 = \mathbb{Z}_j, H^4 = \mathbb{Z}_j, H^5 = \mathbb{Z}$. Once again, $\mathbb{Z}/j\mathbb{Z} \simeq \mathbb{Z}_j$.

5. PROPERTIES OF $R_{3,2}$

We noted in Sec. (1) that from an element of \mathcal{S} we may obtain a *triple* over W consisting of a principal S^1 -bundle $p : E \rightarrow W$ a class $b \in H^2(E, \mathbb{Z})$ and a class $H \in H^3(E, \mathbb{Z})$. As in Sec. (3), the map $W \rightarrow (\mathbf{Triples\ over\ } W)$ is a set-valued functor on the category of unbased CW complexes. It has a classifying space denoted $R_{3,2}$. We have natural fibrations $q_2 : R_{3,2} \rightarrow R_2$ and $q_3 : R_{3,2} \rightarrow R_3$. We will show below that this classifying space has a canonical bundle over it which classifies triples.

We noted above that $R_{3,2}$ is a fiber product $R_2 \times_{K(\mathbb{Z}, 2)} R_3$. As in Sec.

(3), this classifying space is unbased. However, one may choose a basepoint for it as in that section. We pick basepoints $r_i \in R_i$ and a basepoint $r_{3,2} \in R_{3,2}$ such that $q_2(r_{3,2}) = r_2$ and $q_3(r_{3,2}) = r_3$. (That is, the q_i may be taken to be based maps.)

Theorem 5.1. (1) *The homotopy groups of $R_{3,2}$ are*

- $\pi_1(R_{3,2}) \simeq \mathbb{Z}$,
- $\pi_2(R_{3,2}) \simeq \mathbb{Z}^3$,
- $\pi_3(R_{3,2}) \simeq \mathbb{Z}$,

- $\pi_i(R_{3,2}) \simeq 0, i > 3.$
- (2) $R_{3,2}$ is not simple

Proof. Given an unbased map $f : W \rightarrow R_{3,2}$, classifying a triple $([p], b, H)$ over W , we pick a basepoint x_0 for W . We use the argument of Lemma (3.3), to homotope f to a based map. Note that $q_3 \circ f$ is a pair over W , the pair $([p], H)$. If $q_3 \circ f$ is nullhomotopic, then $[p] = 0, H = 0$ and hence b defines classes in $H^2(W \times S^1, \mathbb{Z})$ and $H^1(W \times S^1, \mathbb{Z})$. That is, we obtain a based map $W \rightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$ and hence the homotopy fiber of q_3 is $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$. Similarly we see that the homotopy fiber of q_2 is $K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2)$.

- (1) We know that $R_{3,2}$ may be taken to be a based fibration of the form

$$K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1) \rightarrow R_{3,2} \rightarrow R_3$$

and also a based fibration of the form

$$K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2) \rightarrow R_{3,2} \rightarrow R_2.$$

This gives two long exact sequences for the homotopy groups of $R_{3,2} : 0 \rightarrow \pi_3 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \pi_2 \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z} \rightarrow \pi_1 \rightarrow 0$ and $0 \rightarrow \mathbb{Z} \rightarrow \pi_3 \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow \pi_2 \rightarrow \mathbb{Z}^2 \rightarrow 0 \rightarrow \pi_1 \rightarrow \mathbb{Z}$. From these sequences it is clear that $\pi_i(R_{3,2}) \simeq 0$ if $i > 3$. Further, from the second sequence $\pi_3 \simeq \mathbb{Z}$, and $\pi_2 \simeq \mathbb{Z}^3$. Then it follows that $\pi_1 \simeq \mathbb{Z}$.

- (2) From the long exact sequence of the fibration over R_2 , we have a $\pi_1(R_2)$ equivariant sequence $0 \rightarrow \pi_2(K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2)) \rightarrow \pi_2 \rightarrow \mathbb{Z}^2 \rightarrow 0 \rightarrow \pi_1 \rightarrow \mathbb{Z}$. From this, it is clear that the generator of $\pi_1(R_{3,2})$ maps isomorphically to the generator of $\pi_1(R_2)$. In addition, this generator acts on the degree two part of the sequence. Since $K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2)$ is simply connected and since every action of π_1 factors through the action of the fundamental group, the generator of $\pi_1(R_{3,2})$ acts trivially on $\pi_2(K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2))$. In the above sequence, this generator acts nontrivially on $\pi_2(R_2) \simeq \mathbb{Z}^2$ and hence must act nontrivially on $\pi_2(R_{3,2})$ since the sequence is π_1 -equivariant. Hence $R_{3,2}$ is not simple.

□

We determine the action of $\pi_1(R_{3,2})$ on $\pi_2(R_{3,2})$ below (see Eq. (8) below and surrounding text).

The T-duality map should be a map $T_{3,2} : R_{3,2} \rightarrow R_{3,2}$. It might be hoped that this map is a fiber product of two maps $R_3 \rightarrow R_3$ and $R_2 \rightarrow R_2$. Unfortunately, $T_{3,2}$ cannot be of this form: Note that under

T-duality a pair $([p], b)$ maps into a *triple* $(0, b^\#, [p] \times z)$. To determine T , we first need to study the structure of $R_{3,2}$. Since $R_{3,2}$ is a fibration over R_2 , and R_2 is a mapping torus (see previous section), it might be hoped that $R_{3,2}$ is also a mapping torus. This is indeed the case.

Lemma 5.1. *The universal cover $\tilde{R}_{3,2}$ of $R_{3,2}$ is homotopy equivalent to $R_3 \times K(\mathbb{Z}, 2)$. Hence $R_{3,2}$ is a mapping torus of a map $\psi : R_3 \times K(\mathbb{Z}, 2) \rightarrow R_3 \times K(\mathbb{Z}, 2)$.*

Proof. Let $p : \tilde{R}_{3,2} \rightarrow R_{3,2}$ be the covering projection. Pick $\tilde{r}_{3,2} \in \tilde{R}_{3,2}$ such that $p(\tilde{r}_{3,2}) = r_{3,2}$. Consider the map $\phi_* : \pi_2(\tilde{R}_{3,2}) \rightarrow \pi_2(R_3)$ given by $\phi_* = q_{2*} \circ p_*$. $\pi_2(\tilde{R}_{3,2}) \xrightarrow{p_*} \pi_2(R_{3,2}) \xrightarrow{q_{3*}} \pi_2(R_3)$. Note that p_* is an isomorphism by definition. I claim ϕ is onto: We have a fibration $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1) \rightarrow R_{3,2} \xrightarrow{q_3} R_3$. The long exact sequence of homotopy groups associated to this fibration is

$$(7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\lambda} & \mathbb{Z} & \xrightarrow{\mu} & \mathbb{Z} \xrightarrow{\kappa} \mathbb{Z}^3 \\ & & \xrightarrow{\phi} & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow \mathbb{Z} \longrightarrow 0. \end{array}$$

Note that all the groups in the sequence are torsion-free. Here, λ is an injective map $\mathbb{Z} \rightarrow \mathbb{Z}$ and hence λ is an isomorphism and $\ker(\mu) \simeq \mathbb{Z}$. Hence by exactness and the absence of torsion $\mu = 0$, κ is injective and ϕ onto.

Consider the map $\phi = q_3 \circ p : \tilde{R}_{3,2} \rightarrow R_3$. By the arguments in Lemma (3.3) we may take it to be based. Let W be its homotopy fiber. Then the l.e.s. of the fibration $W \rightarrow \tilde{R}_{3,2} \rightarrow R_3$ is

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \pi_3(W) & \xrightarrow{\alpha} & \mathbb{Z} & \xrightarrow{\beta} & \mathbb{Z} & \xrightarrow{\gamma} & \pi_2(W) & & \\ & & \xrightarrow{\mu} & \mathbb{Z}^3 & \xrightarrow{\kappa} & \mathbb{Z}^2 & \xrightarrow{\nu} & \pi_1(W) & \longrightarrow & 0 & \longrightarrow 0. \end{array}$$

We know κ is onto and hence $\nu = 0$ and $\pi_1(W) = 0$. Since κ is onto, $\ker(\kappa) \simeq \mathbb{Z}$, and hence, $\text{im}(\mu) \simeq \mathbb{Z}$.

Now, consider the map $\phi_* : \pi_3(\tilde{R}_{3,2}) \rightarrow \pi_3(R_3)$ given by $\phi_* = q_{3*} \circ p_*$ in degree 3. Note that p_* is an isomorphism by definition. By the l.e.s. Eq. (7), $\beta = \lambda \circ p_*$. By the argument given above both λ and p_* are isomorphisms, and so is β . Hence, $\text{im}(\alpha) = 0$ and $\pi_3(W) \simeq 0$.

Also $\ker(\gamma) \simeq \mathbb{Z}$ and so $\gamma = 0$. Hence $\pi_2(W) \simeq \mathbb{Z}$ by counting ranks. Its clear that $\pi_i(W) \simeq 0$ if $i > 3$. Therefore, W is homotopy equivalent to $K(\mathbb{Z}, 2)$.

Now, $W \rightarrow \tilde{R}_{3,2} \rightarrow R_3$ is a based fibration with $W \sim K(\mathbb{Z}, 2)$. Also, $R_{3,2}$ and R_3 are spaces of finite type since their fundamental groups

are finitely generated (by Ref. [18]) and the action of $\pi_1(R_3)$ on W is zero since $\pi_1(R_3) \simeq 0$. By Lemma 8^{bis}.28 of Ref. [19], this implies that the fibration is principal. Hence, $\tilde{R}_{3,2} \simeq K(\mathbb{Z}, 2) \times R_3$ because by Ref. [2] $H^3(R_3, \mathbb{Z}) = 0$ and so the classifying map $R_3 \rightarrow K(\mathbb{Z}, 3)$ is always trivial.

Since $R_3 \times K(\mathbb{Z}, 2)$ is homotopy equivalent to the universal cover of $R_{3,2}$, we have a commutative diagram of spaces

$$\begin{array}{ccccc} R_3 \times K(\mathbb{Z}, 2) & \xrightarrow{\tilde{q}_2} & K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) & & \\ p_{3,2} \downarrow & & p_2 \downarrow & & \\ R_{3,2} & \xrightarrow{q_2} & R_2 & \longrightarrow & K(\mathbb{Z}, 1). \end{array}$$

which gives a diagram of homotopy groups and maps equivariant under the action of $\pi_1(K(\mathbb{Z}, 1)) \simeq \mathbb{Z}$. It is clear that since R_2 is the mapping torus of $K(\mathbb{Z}^2, 2)$ under the lift of this action, $R_{3,2}$ is the mapping torus of $R_3 \times K(\mathbb{Z}, 2)$ under the lift of this action (by the commutativity of the diagram). The generator of this action acts on $R_3 \times K(\mathbb{Z}, 2)$ as the generator ψ of the deck transformation group. \square

It is interesting to note that we have a sequence of mapping tori

$$R_{3,2} \rightarrow R_2 \rightarrow K(\mathbb{Z}, 1)$$

such that there are isomorphisms $\pi_1(R_{3,2}) \simeq \pi_1(R_2) \simeq \pi_1(K(\mathbb{Z}, 1))$ and $\pi_1(K(\mathbb{Z}, 1))$ acts on the homotopy of each of these spaces.

Note that \tilde{R}_2 is a trivial $K(\mathbb{Z}, 2)$ fibration over $K(\mathbb{Z}, 2)$ (since R_2 is). Also, $\tilde{R}_3 \simeq R_3$ is a fibration over $K(\mathbb{Z}, 2)$ as well (since R_3 is). It then follows from the definition that $\tilde{R}_{3,2}$ is the fiber product of \tilde{R}_3 and \tilde{R}_2 over $K(\mathbb{Z}, 2)$.

We have a commutative diagram

$$\begin{array}{ccc} R_3 \times K(\mathbb{Z}, 2) & \xrightarrow{\tilde{q}_2} & K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \\ p_{3,2} \downarrow & & p_2 \downarrow \\ R_{3,2} & \xrightarrow{q_2} & R_2. \end{array}$$

We note that $\pi_2(R_{3,2}) \simeq \pi_2(\tilde{R}_{3,2})$. We choose generators for $\pi_2(\tilde{R}_{3,2}) \simeq \mathbb{Z}^3$ such that the projection $(a, b, c) \mapsto c$ is induced by the map $R_3 \times K(\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}, 2)$. Also the projection $(a, b, c) \mapsto (a, b)$ is induced by the map $R_3 \times K(\mathbb{Z}, 2) \rightarrow R_3$. With these choices of generators, the map \tilde{q}_2 is given by $(a, b, c) \mapsto (a, c)$.

In Sec. (3) we had noted that the deck group of \tilde{R}_2 acts on pairs by multiplication by a matrix of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The map \tilde{q}_2^* on π_2 must be equivariant under the action of the deck group since q_2^* is equivariant under the action of π_1 and the maps $p_{3,2}^*$ and p_2^* are isomorphisms on π_2 . This implies that the deck group acts on $\pi_2(\tilde{R}_{3,2})$ by multiplication by the matrix

$$(8) \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the isomorphisms $p_{3,2}^*$ and p_2^* above, this is also the action of $\pi_1(R_{3,2})$ on $\pi_2(R_{3,2})$. We may also determine the deck transformation explicitly as follows:

Lemma 5.2. *The deck transformation $\psi : \tilde{R}_{3,2} \rightarrow \tilde{R}_{3,2}$ has the form $\psi = \pi_1 \times f$ where $\pi_1 : R_3 \times K(\mathbb{Z}, 2) \rightarrow R_3$ is the projection onto the first factor and $f : R_3 \times K(\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}, 2)$ is determined below.*

Proof. Consider a triple over a simply connected space W . This determines a classifying map $f : W \rightarrow R_{3,2}$. Using the argument of Lemma (3.3) we may take it to be based. Since W is simply connected, this map lifts to $\tilde{R}_{3,2}$. This implies that a triple (p, b, H) over W is determined by a pair $([p], H)$, $p : E \rightarrow W$, and a class b in $H^2(W, \mathbb{Z})$. From the Gysin sequence, since $H^1(W, \mathbb{Z}) \simeq 0$ we see that $\mathbb{Z} \simeq H^0(W, \mathbb{Z}) \xrightarrow{\cup p} H^2(W, \mathbb{Z}) \rightarrow H^2(E, \mathbb{Z}) \rightarrow 0$ is exact. Thus, every class in $H^2(E, \mathbb{Z})$ is the image of some class in $H^2(W, \mathbb{Z})$ from the Gysin sequence. This is the class b above. Further two such classes differ by an integral multiple of p .

It is also clear that if $\psi : \tilde{R}_{3,2} \rightarrow \tilde{R}_{3,2}$ is a deck transformation, then, since $p_{3,2} \circ \psi = \psi$, $\psi \circ f$ represents the same triple over W . Thus, $\psi \circ f$ is also representable as $((p, H), b')$ for some b' . If \tilde{f} is the lift of a map $f : W \rightarrow R_{3,2}$ representing a triple (p, b, H) to $\tilde{R}_{3,2}$ then it is clear that all possible lifts may be represented as $((p, H), y + mp)$, $m \in \mathbb{Z}$ where $y \in H^2(W, \mathbb{Z})$ and $p^*(y) = b$. By the above, the action of the deck transformation is to shift $((p, H), y)$ to $((p, H), y + mp)$. Thus, $\psi = \pi_1 \times f$ where $f : R_3 \times K(\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}, 2)$ is the map defined by $((p, H), y) \rightarrow y + mp$, $m \in \mathbb{Z}$, and $\pi_1 : R_3 \times K(\mathbb{Z}, 2) \rightarrow R_3$ be the projection onto the first factor. \square

It is clear that the map f above defines a class in $H^2(R_3 \times K(\mathbb{Z}, 2), \mathbb{Z})$.

As an illustration of this, consider all triples over S^2 . For any principal circle bundle $E_p \rightarrow S^2$, we have that $H^3(E_p, \mathbb{Z}) \simeq \mathbb{Z}$. Hence, for each pair $([p], b)$, the H -flux can have countably many values.

Lemma 5.3. *The set of unbased homotopy classes of maps $S^2 \rightarrow R_3 \times K(\mathbb{Z}, 2)$ modulo the action of the the deck transformation group on $\tilde{R}_{3,2}$ is exactly the set of triples over S^2 .*

Proof. Clearly, the set of unbased homotopy classes of maps $S^2 \rightarrow R_3 \times K(\mathbb{Z}, 2)$ may be made based since both R_3 and $K(\mathbb{Z}, 2)$ are connected and simply connected: This set is exactly $\pi_2(R_3 \times K(\mathbb{Z}, 2))$.

Hence, this set modulo the action of the the deck transformation group on $\tilde{R}_{3,2}$ is the same as $\pi_2(\tilde{R}_{3,2})$ modulo the action of $\pi_1(\tilde{R}_{3,2})$. By Thm. 5.1, this is the quotient of \mathbb{Z}^3 under the action of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From the previous section (in particular the discussion before and after Thm. 3.9) we know that the quotient of \mathbb{Z}^2 by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is exactly the collection of pairs over S^2 . The action on \mathbb{Z}^3 leaves the last coordinate fixed. Hence, for each isomorphism class of pairs $([p], b)$ over S^2 , we obtain a countable number of triples as required. \square

Note that at least for S^2 , the only action of the deck group is to shift a pair $([p], H)$ to an equivalent pair.

6. THE T-DUALITY MAPPING

We note that the maps $q_2 : R_{3,2} \rightarrow R_2$ and $q_3 : R_{3,2} \rightarrow R_3$ have natural sections. Thus these maps are injective on cohomology. In particular (from the Theorem below) $H^2(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}^2 \simeq H^2(R_3, \mathbb{Z}) \simeq \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$. Similarly, $H^2(R_2, \mathbb{Z}) \hookrightarrow H^2(R_{3,2}, \mathbb{Z})$. The latter map may be taken as the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}$ into the first factor. Thus we may write $a_1 = q_2^*(b)$. Further, from the Theorem below, we see that $H^1(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}l$. Therefore $l = q_2^*(a)$ and so $a_1 \cdot l = 0$ since $b \cdot a = 0$ in $H^*(R_2, \mathbb{Z})$ (See Eq. (5)).

In addition since $R_{3,2} \rightarrow R_3$ is fiber-preserving over $K(\mathbb{Z}, 2)$, and the natural map $R_{3,2} \rightarrow K(\mathbb{Z}, 2)$ has sections, we see that $\mathbb{Z}[c] \simeq H^*(K(\mathbb{Z}, 2), \mathbb{Z}) \hookrightarrow H^*(R_{3,2}, \mathbb{Z})$.

Theorem 6.1. *The cohomology groups of $R_{3,2}$ are*

- $H^0(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^1(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}l$,
- $H^2(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$,
- $H^3(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}a_2l$,
- $H^4(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}x$

where $a_1 \cdot a_2 = 0$ and $a_1 \cdot l = 0$. Also, $a_1 = q_2^*(b)$.

Proof. $R_{3,2}$ is connected, being a fiber product of two connected spaces over a connected space. Hence $H^0(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$. We know that the universal cover of $R_{3,2}$ is $R_3 \times K(\mathbb{Z}, 2)$ with deck transformation group \mathbb{Z} . Note that by the Hurewicz theorem, $H^2(\tilde{R}_{3,2}, \mathbb{Z}) \simeq \pi_2(\tilde{R}_{3,2})$. Further, since $\tilde{R}_{3,2}$ is the universal cover, $\pi_2(\tilde{R}_{3,2}) \simeq \pi_2(R_{3,2}) \simeq \mathbb{Z}^3$. We calculated the action of $\pi_1(R_{3,2})$ on $\pi_2(R_{3,2})$ above. Using this we see that the action of $\pi_1(R_{3,2})$ on $H^2(\tilde{R}_{3,2}, \mathbb{Z})$ is multiplication by the same matrix. The cohomology ring of $\tilde{R}_{3,2}$ is given by

- $H^0(\tilde{R}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^1(\tilde{R}_{3,2}, \mathbb{Z}) \simeq 0$,
- $H^2(\tilde{R}_{3,2}, \mathbb{Z}) \simeq (\mathbb{Z}a_1 \oplus \mathbb{Z}a_2) \oplus \mathbb{Z}c$,
- $H^3(\tilde{R}_{3,2}, \mathbb{Z}) \simeq 0$,
- $H^4(\tilde{R}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}a_1c \oplus \mathbb{Z}a_2c \oplus \mathbb{Z}c^2$,

The action of the deck group on H^2 may be used to calculate the action of the deck group on H^4 . This is multiplication by the matrix (in the basis $(a_1^2, a_2^2, a_1c, a_2c, c^2)$.)

$$\phi = \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

We apply the Cartan-Leray spectral sequence to this universal cover to compute the cohomology groups of $R_{3,2}$. Since \mathbb{Z} is cohomologically one dimensional, the sequence collapses at the E_2 page itself. In particular only $E_2^{q,1}$ and $E_2^{q,2}$ are nonzero. We find that $H^0(\mathbb{Z}, H^q(\tilde{R}_{3,2}, \mathbb{Z})) \simeq H^q(\tilde{R}_{3,2}, \mathbb{Z})^{\mathbb{Z}}$ and $H^1(\mathbb{Z}, H^q(\tilde{R}_{3,2}, \mathbb{Z})) \simeq H^q(\tilde{R}_{3,2}, \mathbb{Z})/(\phi - 1)H^q(\tilde{R}_{3,2}, \mathbb{Z})$. Using this action we obtain the cohomology groups shown.

Here a_i are the pullbacks of the generator of $H^2(R_3, \mathbb{Z})$ and so $a_1 \cdot a_2 = 0$. In addition l is the pullback of the generator a of $H^1(R_2, \mathbb{Z})$ via q_2^* . Also, $a_1 = q_2^*(b)$ and so¹² $a_1 \cdot l = q_2^*(b \cdot a) = 0$. Also, x is a new generator in degree 3. \square

¹²See Eq. (5).

Let $p_3 : E_3 \rightarrow R_3$ be the classifying bundle E of Ref. [2] Sec. (2.4). By Sec. (3), there is a classifying bundle $p_2 : E_2 \rightarrow R_2$. By the isomorphisms on H^2 discussed above, it is clear that $q_2^*E_2 \simeq q_3^*E_3$ and we denote this bundle by $E_{3,2}$. This bundle has characteristic class $a_1 \in H^2(R_{3,2}, \mathbb{Z})$. Let $p : E_{3,2} \rightarrow R_{3,2}$ be the bundle map.

Theorem 6.2. *The cohomology groups of $E_{3,2}$ are*

- $H^0(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^1(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}y$,
- $H^2(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}p^*(a_2) \oplus \mathbb{Z}b$,
- $H^3(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}p^*(a_2l) \oplus \mathbb{Z}h$.

Here $y = p^*(l)$ so $p_!(y) = 0$, $p_!(b) = l$ and $p_!(h) = a_2$.

Proof. Consider the Gysin sequences associated to $E_{3,2}$, E_3 and E_2 . By naturality, there are morphisms from the sequences associated to E_3 and E_2 to the sequence associated to $E_{3,2}$ induced by the maps q_2, q_3 . Further, the maps q_i have natural sections and hence q_i^* are injective on cohomology.

- Degree 0 $E_{3,2}$ is a fibration of a connected space (S^1) over a connected base ($R_{3,2}$). Hence it is connected and $H^0(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$.
- Degree 1 From the Serre spectral sequence, we find that $H^1(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}y$. From the Gysin sequence for $p : E_{3,2} \rightarrow R_{3,2}$ we have

$$\mathbb{Z}l \xrightarrow{p^*} \mathbb{Z}y \xrightarrow{p_!} \mathbb{Z}.$$

Now, since p^* is injective, by exactness, and the absence of torsion $y = p^*(l)$ and $p_!(y) = 0$.

- Degree 2 From the Gysin sequence associated to $p : E_{3,2} \rightarrow R_{3,2}$ beginning at $H^0(R_{3,2}, \mathbb{Z})$ we have

$$\mathbb{Z} \xrightarrow{\phi_1 = \cup a_1} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{p^*} H^2(E_{3,2}, \mathbb{Z}) \xrightarrow{p_!} \mathbb{Z}l \xrightarrow{\cup a_1} \mathbb{Z}a_2l$$

From the previous part of the sequence ϕ_1 is injective. Therefore, by exactness $H^2(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}p^*(a_2) \oplus \mathbb{Z}b$. Note that the generator w of $H^2(E_2, \mathbb{Z})$ pulls back to $H^2(E_{3,2}, \mathbb{Z})$ as a generator b . Then, we have $p_!(b) = l \in H^1(R_{3,2}, \mathbb{Z})$. Under pullback by a classifying map $W \rightarrow R_{3,2}$, b pulls back to the B -class of the triple $([p], b, H)$ being classified.

- Degree 3

Consider the Gysin sequence for $p : E_{3,2} \rightarrow R_{3,2}$

$$\mathbb{Z}l \xrightarrow{\cup a_1} \mathbb{Z}a_2l \xrightarrow{p^*} H^3(E_{3,2}, \mathbb{Z}) \xrightarrow{p_!} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{\cup a_1} \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}x.$$

We have that $a_1 \cdot l = 0$ (See Eq. (5)) hence p^* is injective, also the last map has kernel $\mathbb{Z}a_2 \subset \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$. By exactness, $H^3(E_{3,2}, \mathbb{Z}) \simeq p^*(a_2l) \oplus \mathbb{Z}h$. Note that $H^3(E_{3,2}, \mathbb{Z})$ contains a \mathbb{Z} -subgroup which is $\text{im}(\lambda_2) \simeq H^3(E_3, \mathbb{Z}) \simeq h\mathbb{Z}$ such that $p_!(h) = a_2$. This class pulls back along the classifying map $W \rightarrow R_{3,2}$ to the H -flux of the triple being classified.

□

Thus over $R_{3,2}$ we have the *canonical triple* $([p], b, h)$ corresponding to the bundle $E_{3,2}$ above.

Lemma 6.1. *The pullback of the fibration $q_3 : R_{3,2} \rightarrow R_3$ along the T-duality map $T_3 : R_3 \rightarrow R_3$ is $R_{3,2}$.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} T_{3,2}^* R_{3,2} & \xrightarrow{T_{3,2}} & R_{3,2} \\ q_3 \downarrow & & \downarrow q_3 \\ R_3 & \xrightarrow{T_3} & R_3 \end{array}$$

where we define $T_{3,2}$ as the map on $R_{3,2}$ induced by the pullback and, by definition,

$$T_{3,2}^* R_{3,2} = \{(x, y) \in R_3 \times R_{3,2} | T_3(x) = q_3(y)\}.$$

Given $f : W \rightarrow T_{3,2}^* R_{3,2}$, $f(w) = (f_1(w), f_2(w))$ such that $T_3 \circ f_1 = q_3 \circ f_2$, we get $T_3 \circ f_1 : W \rightarrow R_3$ and $q_2 \circ f_2 : W \rightarrow R_2$. Obviously these maps define pairs $([p], b)$ and $([p], H)$ with the *same* $[p]$. Conversely, given two such pairs, we obtain maps $h_1 : W \rightarrow R_3$, $h_2 : W \rightarrow R_2$, such that $\mu \circ h_1 = \mu \circ h_2$ where $\mu : R_i \rightarrow K(\mathbb{Z}, 2)$ is the map sending a pair to the class of the principal bundle in the pair. We define $g_1 = T_3^{-1} \circ h_1 : W \rightarrow R_3$. By the universal property of the fiber product, there is a map $g_2 : W \rightarrow R_{3,2}$. Then $q_3 \circ g_2 = h_1 = T_3 \circ g_1$ by definition. Hence, (g_1, g_2) give a well-defined map $W \rightarrow T_{3,2}^* R_{3,2}$. Therefore $T_{3,2}^* R_{3,2}$ also classifies triples of the form $([p], b, H)$ since a 2-pair and a 3-pair with the same $[p]$ define and are defined by the same triple. Hence, by uniqueness of the classifying space of triples, we get a homeomorphism $\phi : T_{3,2}^* R_{3,2} \rightarrow R_{3,2}$. □

Note that if X is a CW-complex, and $f : X \rightarrow R_{3,2}$ is a classifying map which corresponds a triple (p, b, H) on X the pair associated to this triple is (p, H) . The triple associated to $T_{3,2} \circ f$ is $(p^\#, b^\#, H^\#)$ and the associated pair is $(p^\#, b^\#)$. Thus, we see that $q_2 \circ T_{3,2} \neq q_2$. However, we always require $q_2 = q_2 \circ \phi$, i.e., we require that $a_1 = q_2^*(b)$

in $H^*(R_{3,2}, \mathbb{Z})$ and in $H^*(T^*R_{3,2}, \mathbb{Z})$ as well. In addition we require that $a_1 \cdot l = a_1 \cdot a_2 = 0$ with no such relation on a_2 in both spaces.

Note that $q_3 \circ T_{3,2} = T_3 \circ q_3$ and hence $T_{3,2}^*$ acts on $H^2(R_{3,2}, \mathbb{Z})$ by interchanging a_1 and a_2 .

Lemma 6.2. *Let $\hat{E}_{3,2}$ be the bundle over $R_{3,2}$ of characteristic class $a_2 \in H^2(R_{3,2}, \mathbb{Z})$. The cohomology groups of $\hat{E}_{3,2}$ are*

- $H^0(\hat{E}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^1(\hat{E}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}\hat{y}$,
- $H^2(\hat{E}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}p^*(a_1)$,
- $H^3(\hat{E}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}\hat{h}$,

and $\hat{p}_!(\hat{y}) = 0, \hat{y} = \hat{p}^*(l), \hat{p}_!(\hat{h}) = a_1$.

Proof. The calculation for H^0 and H^1 is exactly the same as for $H^0(E_{3,2}, \mathbb{Z})$ and $H^1(E_{3,2}, \mathbb{Z})$ with y replaced by \hat{y} .

- Degree 2 Consider the Gysin sequence for $\hat{E}_{3,2}$ beginning at $H^0(R_{3,2}, \mathbb{Z})$

$$\mathbb{Z} \xrightarrow{\phi_1 = \cup a_2} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{\hat{p}^*} H^2(E_{3,2}, \mathbb{Z}) \xrightarrow{\hat{p}_!} \mathbb{Z}l \xrightarrow{\cup a_2} \mathbb{Z}a_2l$$

The map ϕ_1 is injective because of the previous part of the sequence. The last map is injective and surjective since $a_2 \cdot l \neq 0$ because we chose $q_2 = q_2 \circ \phi$ above¹³. Hence, by exactness, the map $\hat{p}_!$ is zero and $H^2(\hat{E}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}p^*(a_1)$.

- Degree 3 Consider the Gysin sequence for $\hat{E}_{3,2}$ beginning at $H^1(R_{3,2}, \mathbb{Z})$

$$\mathbb{Z}l \xrightarrow{\phi_1 = \cup a_2} \mathbb{Z}a_2l \xrightarrow{\hat{p}^*} H^3(E_{3,2}, \mathbb{Z}) \xrightarrow{\hat{p}_!} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{\cup a_2} \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}x$$

The map ϕ_1 is an isomorphism and hence $\hat{p}^* = 0$. By exactness, $H^3(\hat{E}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}\hat{h}$ and $\hat{p}_!(\hat{h}) = a_1$.

□

Theorem 6.3. (1) *The bundle $\hat{p} : \hat{E}_{3,2} \rightarrow R_{3,2}$ described above and cohomology classes $\hat{b} \in H^2(\hat{E}_{3,2}, \mathbb{Z})$ and $\hat{h} \in H^2(\hat{E}_{3,2}, \mathbb{Z})$ are such that $T_{3,2}$ classifies the triple $([\hat{p}], \hat{b}, \hat{h})$ over $R_{3,2}$.*

(2) *Let $\tilde{T} : \hat{E}_{3,2} \rightarrow E_{3,2}$ be the map covering $T_{3,2} : R_{3,2} \rightarrow R_{3,2}$, then \tilde{T} acts on the generators of the low-dimensional cohomology of $H^*(E_{3,2}, \mathbb{Z})$ as follows*

- $\tilde{T}^*(y) = 0$
- $\tilde{T}^*(p^*(a_2)) = \hat{p}^*(a_1)$

¹³See paragraph after the proof of the previous Lemma.

- $\tilde{T}^*(p^*(a_2l)) = 0$
 - $\tilde{T}^*(h) = \hat{h}$
 - The T -dual of b is of the form $kp^*(a_1)$ for some k . Hence, the T -dual of a triple (p, b, H) is a triple of the form $(\hat{p}, p^*(y), \hat{H})$.
- (3) $T_{3,2}^2$ cannot be homotopic to the identity, i.e. T -duality for triples is not involutive.

Proof. (1) We consider the triple $([\hat{p}], \hat{b}, \hat{h})$ over $R_{3,2}$. This triple is classified by a map $f : R_{3,2} \rightarrow R_{3,2}$. Then $f^*(a_1) = a_2$. It is clear that $f^*E_{3,2} = \hat{E}_{3,2}$. In addition if we forget the B -class, the action of f on the pair $([p], h)$ is to transform it into $([\hat{p}], \hat{h})$. This implies that $q_3 \circ f = T_3 \circ q_3$. Hence f is homotopic to $T_{3,2}$ by the uniqueness of the pullback square described in Thm. (6.1)

- (2) Now, in the above two sequences, $T_{3,2}^*$ induces a natural map $(a_1, a_2) \rightarrow (a_2, a_1)$ on $H^2(R_{3,2}, \mathbb{Z})$, further the induced maps on the remaining cohomology groups except $H^2(E_{3,2}, \mathbb{Z})$ are the identity.

We have that $\hat{p}^* \circ T_{3,2}^* = \tilde{T}^* \circ p^*$. As a result, $\tilde{T}^*(p^*(a_2)) = \hat{p}^*(a_1)$. Also, we have $T_{3,2}^* \circ p_! = \hat{p}_! \circ \tilde{T}^*$. Hence $\hat{p}_! \circ \tilde{T}^*(b) = T_{3,2}^*(l)$. However, from the Gysin sequence for $\hat{E}_{3,2}$, we have

$$\mathbb{Z}\hat{p}^*(a_1) \xrightarrow{\hat{p}_!} \mathbb{Z}l \xrightarrow{\cup a_2} \mathbb{Z}a_2l \rightarrow \mathbb{Z}\hat{h}.$$

Now the map $H^1(R_{3,2}, \mathbb{Z}) \rightarrow H^3(R_{3,2}, \mathbb{Z})$ is an isomorphism and hence $\hat{p}_!$ is zero. As a result, $T_{3,2}^*(l) = 0$, and hence $\tilde{T}^*(y) = 0$.

Consider the above pair of Gysin sequences beginning at $H^3(R_{3,2}, \mathbb{Z})$: We have for $E_{3,2}$

$$\mathbb{Z}l \xrightarrow{\cup a_1} \mathbb{Z}a_2l \xrightarrow{p^*} \mathbb{Z}p^*(a_2l) \oplus \mathbb{Z}h \xrightarrow{p_!} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{\cup a_1} \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}x$$

and for $\hat{E}_{3,2}$

$$\mathbb{Z}l \xrightarrow{\cup a_2} \mathbb{Z}a_2l \xrightarrow{\hat{p}^*} \mathbb{Z}\hat{h} \xrightarrow{\hat{p}_!} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{\cup a_2} \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}x$$

From the above action of $T_{3,2}^*$ on $H^2(R_{3,2}, \mathbb{Z})$ it is clear that $\tilde{T}^*(h) = \hat{h}$. Also, since p^* is an isomorphism and \hat{p}^* is zero, it is clear that $\tilde{T}^*(p^*(a_2l)) = 0$. From the above, we have

- $\tilde{T}^*(y) = 0$
- $\tilde{T}^*(p^*(a_2)) = \hat{p}^*(a_1)$
- $\tilde{T}^*(p^*(a_2l)) = 0$
- $\tilde{T}^*(h) = \hat{h}$.

- This is clear since, by the above, we know that $\hat{p}_!(\tilde{T}(b)) = 0$. The result for the form of the T-dual triple is obvious.
- (3) From the above, it is clear that
 - $T_{3,2}^*(l) = 0$
 - $T_{3,2}^*(a_1, 0) = (0, a_2)$ and $T_{3,2}^*(0, a_2) = (a_1, 0)$.
 - $T_{3,2}^*(a_2 l) = 0$
 Therefore $T_{3,2}^2$ cannot be homotopic to the identity.

□

Theorem 6.4. (1) *There is a natural map $\gamma : R_3 \rightarrow R_{3,2}$ and hence there is a natural triple associated to a pair (p, H) over any space, namely the triple $(p, p^*(p_!H), H)$.*

(2) *The T-dual of the triple in the previous item is the triple $(\hat{p}, \hat{p}^*(\hat{p}_!\hat{H}), \hat{H})$, that is, $T_{3,2} \circ \gamma = \gamma \circ T_3$.*

(3) *Each class α in $H^2(K(\mathbb{Z}^2, 2), \mathbb{Z})$ gives a map $\gamma_\alpha : R_3 \rightarrow R_{3,2}$. Each of these maps is an inverse to the forgetful map $R_{3,2} \rightarrow R_3$ which ‘forgets’ the class b .*

Proof. (1) Note that we have a natural map $R_3 \rightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2)$ which is given by sending $(p, H) \rightarrow (p, p_!(H))$. We have a natural identification (up to homotopy) of $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2)$ with \tilde{R}_2 and, as a result, there is a natural map $\gamma : R_3 \rightarrow R_2$. We may naturally associate to any 3-pair (p, H) over a space X a 2-pair $(p, p^*p_!H)$ over that space. Note that the map γ commutes with the natural maps $\mu : R_3 \rightarrow K(\mathbb{Z}, 2)$ and $\nu : R_2 \rightarrow K(\mathbb{Z}, 2)$. Hence, there is a natural lift η of γ to $R_{3,2}$ given by $\eta : R_3 \rightarrow R_{3,2}$ such that $\eta(a) = (a, \gamma(a)) \in R_3 \times R_2$. This gives a natural triple associated to any pair $(p, H) : \text{To } (p, H) \text{ we associate } (p, p^*p_!H, H)$.

(2) We know that under T-duality, the bundle E_3 on R_3 is mapped to the bundle \hat{E}_3 on R_3 . If we examine the cohomology of these bundles, we see that the T-dual of the triple $(p, p^*p_!h, h)$ should be a triple of the form $(q, k.q^*q_!\hat{h}, \hat{h}), k \in \mathbb{Z}$. By the above theorem, the T-dual is exactly the triple $(q, q^*q_!\hat{h}, \hat{h})$. This implies that the map γ commutes with $T_{3,2}$ and T_3 : That is, $T_{3,2} \circ \gamma = \gamma \circ T_3$.

(3) Note that each map $f : K(\mathbb{Z}^2, 2) \rightarrow K(\mathbb{Z}^2, 2)$ up to homotopy gives a lift of the form above, if the map is such that $f(a, b) = (a, \alpha(a, b))$. Thus, each element α in $H^2(K(\mathbb{Z}^2, 2), \mathbb{Z})$ gives such a lift. It is clear that each of these lifts correspond to triples of the form $(p, k.p^*(a_1), h), k \in \mathbb{Z}$ on R_3 . Note that each of these

lifts is an inverse to the forgetful map $R_{3,2} \rightarrow R_3$ which ‘forgets’ the element b of a triple.

□

We may now prove the result we conjectured in Sec. (4):

Corollary 6.1. *Let W be a CW-complex and $p : E \rightarrow W$ a principal S^1 -bundle over W . Let $b \in H^2(E, \mathbb{Z})$ and $H \in H^3(E, \mathbb{Z})$ be the B-class and H -flux on the bundle. Let $([q], b^\#, H^\#)$ be the T-dual triple. Topological T-duality induces a bijection between the sets*

$$\{b + lp^*p_!(H)\} \text{ and } \{b^\# + mq^*q_!(H^\#)\}.$$

Proof. Consider the universal bundle $E_{3,2}$ over $R_{3,2}$ and the T-dual bundle $\hat{E}_{3,2}$ over $R_{3,2}$. The result is obvious for the universal bundle since $b^\#$ is always of the form $k \cdot q^*q_!(H^\#)$, $k \in \mathbb{Z}$. By pullback, the result is true for X . □

We now return to the question asked at the end of Section (1). By Lemma (1.1) and Section (2), for a given space X , at fixed H this is equivalent to knowing the image of the map $T_{3,2}$ studied above. The above Corollary indicates that all automorphisms with Phillips-Raeburn invariant in a given coset map into automorphisms with Phillips-Raeburn invariant in another coset. This constrains the map and determines it in a large variety of cases as was seen previously in Sec. (4).

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